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Editor:

Dr. Zhang Wenpeng
Department of Mathematics
Northwest University
Xi'an, Shaanxi, P.R.China

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Two problems related to the Smarandache function

Wenpeng Zhang† and Ling Li† ‡

† Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China
‡ Basic Department, Shaanxi Polytechnic Institute, Xianyang, Shaanxi, P.R.China

Abstract For any positive integer n, the famous pseudo Smarandache function $Z(n)$ is defined as the smallest positive integer m such that $n \mid \frac{m(m+1)}{2}$. That is, $Z(n) = \min\left\{m : \ n \mid \frac{m(m+1)}{2}, \ n \in N\right\}$. The Smarandache reciprocal function $S_c(n)$ is defined as $S_c(n) = \max\{m : \ y \mid n! \text{ for all } 1 \leq y \leq m, \text{ and } \ m+1 \nmid n!\}$. That is, $S_c(n)$ is the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$. The main purpose of this paper is to study the solvability of some equations involving the pseudo Smarandache function $Z(n)$ and the Smarandache reciprocal function $S_c(n)$, and propose some interesting conjectures.

Keywords The pseudo Smarandache function, the Smarandache reciprocal function, the dual function, equation, positive integer solutions, conjecture.

§1. Introduction and results

For any positive integer n, the famous pseudo Smarandache function $Z(n)$ is defined as the smallest positive integer m such that $n \mid \frac{m(m+1)}{2}$. That is,

$$Z(n) = \min\left\{m : \ n \mid \frac{m(m+1)}{2}, \ n \in N\right\}.$$

Its dual function $Z^*(n)$ is defined as $Z^*(n) = \max\left\{m : \frac{m(m+1)}{2} \mid n, \ m \in N\right\}$, where N denotes the set of all positive integers. From the definition of $Z(n)$ we can find that the first few values of $Z(n)$ are: $Z(1) = 1$, $Z(2) = 3$, $Z(3) = 2$, $Z(4) = 7$, $Z(5) = 4$, $Z(6) = 3$, $Z(7) = 6$, $Z(8) = 15$, $Z(9) = 8$, $Z(10) = 4$, $Z(11) = 10$, $Z(12) = 8$, $Z(13) = 12$, $Z(14) = 7$, $Z(15) = 5$, $Z(16) = 31$, $\cdots\cdots$. About the elementary properties of $Z(n)$, many authors had studied it, and obtained some interesting results, see references [1], [2], [3], [4], [5] and [6]. For example, the first author [6] studied the solvability of the equations:

$$Z(n) = S(n) \quad \text{and} \quad Z(n) + 1 = S(n),$$

and obtained their all positive integer solutions, where $S(n)$ is the Smarandache function.

On the other hand, in reference [7], A.Murthy introduced another function $S_c(n)$, which called the Smarandache reciprocal function. It is defined as the largest positive integer m such

m, and $m+1 \nmid n!\}$. For example, the first few values of $S_c(n)$ are:

$$S_c(1)=1,\ S_c(2)=2,\ S_c(3)=3,\ S_c(4)=4,\ S_(5)=6,\ S_c(6)=6,\ S_c(7)=10,$$
$$S_c(8)=10,\ S_c(9)=10,\ S_c(10)=10,\ S_c(11)=12,\ S_c(12)=12,\ S_c(13)=16,$$
$$S_c(14)=16,\ S_5(15)=16,\ S_c(16)=16,\ S_c(17)=18,\ S_c(18)=18,\ \cdots\cdots .$$

A.Murthy [7], Ding Liping [8] and Ren Zhibin [9] also studied the elementary properties of $S_c(n)$, and obtained some interesting conclusions, one of them is that if $S_c(n)=x$ and $n\neq 3$, then $x+1$ is the smallest prime greater than n.

The main purpose of this paper is to study the solvability of some equations related to the Smarandache function, and propose some interesting problems. That is, we have the following:

Unsolved problem 1. Whether there exist infinite positive integers n such that the equation

$$S_c(n)+Z(n)=2n. \tag{1}$$

Unsolved problem 2. Find all positive integer solutions of the equation

$$S_c(n)=Z^*(n)+n. \tag{2}$$

§2. Some results on these unsolved problems

In this section, we shall give some new progress on these unsolved problems. First for the problem 1, it is clear that $n=1$ satisfy the equation (1). $n=3$ does not satisfy the equation (1). If $p\geq 5$ and $p^\alpha+2$ are two odd primes, then $n=p^\alpha$ satisfy the equation (1). In fact this time, we have $Z(p^\alpha)=p^\alpha-1$, $S_c(p^\alpha)=p^\alpha+1$. Therefore, $S_c(p^\alpha)+Z(p^\alpha)=p^\alpha+1+p^\alpha-1=2\cdot p^\alpha$. So $n=p^\alpha$ satisfy the equation (1). For example, $n=1$, 5, 11, 17, 29 and 41 are six solutions of the equation (1). We think that the equation (1) has infinite positive integer solutions. Even more, we have the following:

Conjecture 1. For any positive integer n, the equation

$$S_c(n)+Z(n)=2n$$

holds if and only if $n=1$, 3^α and $p^{2\beta+1}$, where $\alpha\geq 2$ be any integer such that $3^\alpha+2$ be a prime, $p\geq 5$ be any prime, $\beta\geq 0$ be any integer such that $p^{2\beta+1}+2$ be a prime.

For the problem 2, it is clear that $n=3$ does not satisfy the equation (2). If $p\geq 5$ be a prime, $n=p^{2\alpha+1}$ such that $n+2$ be a prime, then $S_c(n)=n+1$, $Z^*(n)=1$, so $S_c(n)=Z^*(n)+n$. Therefore, $n=p^{2\alpha+1}$ satisfy the equation (2). Besides these, whether there exist any other positive integer n satisfying the equation (2) is an open problem. We believe that the following conjecture is true.

Conjecture 2. For any positive integer n, the equation

$$S_c(n)=Z^*(n)+n$$

holds if and only if $n=p^{2\alpha+1}$, where $p\geq 5$ be a prime, $\alpha\geq 0$ be any integer such that $p^{2\alpha+1}+2$ be a prime.

In our conjectures, if prime $p \geq 5$, then $p^{2\beta} + 2$ can be divided by 3. So if $p^{\alpha} + 2$ be a prime, then α must be an odd number.

From our conjectures we also know that there exists close relationship between the solutions of the equations (1), (2) and the twin primes. So we think that the above unsolved problems are very interesting and important.

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On the additive k-power complements

Yanchun Guo

Department of Mathematics, Xianyang Normal University,
Xianyang, Shaanxi, P.R.China

Abstract For any positive integer n, let $b_k(n)$ denotes the additive k-power complements of n. That is, $b_k(n)$ denotes the smallest non-negative integer m such that $n+m$ is a perfect k-power. The main purpose of this paper is using the elementary method to study the mean value properties of $\Omega(n+b_k(n))$, and give a sharper asymptotic formula for it, where $\Omega(n)$ denotes the number of all prime divisors of n.

Keywords Additive k-power complements, function of prime divisors, asymptotic formula.

§1. Introduction and result

For any positive integer $n \geq 2$, let $a_k(n)$ denotes the k-power complement sequence. That is, $a_k(n)$ denotes the smallest integer such that $na_k(n)$ is a perfect k-power. In problem 29 of reference [1], Professor F. Smarandache asked us to study the properties of this sequence. About this problem, many people had studied it, and obtained a series results. For example, Yao Weili [2] studied the mean value properties of $d(n \cdot a_k(n))$, and proved that for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} d(na_k(n)) = x\left(A_0 \ln^k x + A_1 \ln^{k-1} x + \cdots + A_{k-1} \ln x + A_k\right) + \mathrm{O}\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $d(n)$ is the Dirichlet divisor function, $A_0,\ A_1,\ \cdots,\ A_k$ are computable constants, ε denotes any fixed positive number.

Similarly, we define the additive k-power complements as follows: for any positive integer n, $b_k(n)$ denotes the smallest non-negative integer such that $n+b_k(n)$ is a perfect k-power. About the elementary properties of $b_k(n)$, some scholars have studied it, and got some useful results. For example, Xu Zhefeng [3] studied the mean value properties of $b_k(n)$ and $d(b_k(n))$, and obtained two interesting asymptotic formulas. That is, for any real number $x \geqslant 3$, we have the asymptotic formulas

$$\sum_{n \leq x} b_k(n) = \frac{k^2}{4k-2} x^{2-\frac{1}{k}} + \mathrm{O}\left(x^{2-\frac{2}{k}}\right),$$

$$\sum_{n \leq x} d(b_k(n)) = \left(1-\frac{1}{k}\right) x \ln x + \left(2\gamma + \ln k - 2 + \frac{1}{k}\right) x + \mathrm{O}\left(x^{1-\frac{1}{k}} \ln x\right),$$

where γ is the Euler constant.

In this paper, we use the elementary and analytic methods to study the mean value properties of $\Omega(n+b_k(n))$, and give a sharper asymptotic formula for it, where $\Omega(n)$ denotes the number of all prime divisors of n, i.e., $\Omega(n)=\alpha_1+\alpha_2+\cdots+\alpha_r$, if $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the factorization of n into prime powers. That is, we shall prove the following:

Theorem. For any real number $x\geq 2$, we have the asymptotic formula

$$\sum_{n\leq x}\Omega(n+b_k(n))=kx\ln\ln x+k(A-\ln k)x+\mathrm{O}\left(\frac{x}{\ln x}\right),$$

where $A=\gamma+\sum\limits_{p}\left(\ln\left(1-\frac{1}{p}\right)+\frac{1}{p-1}\right)$ be a constant, $\sum\limits_{p}$ denotes the summation over all primes, and γ be the Euler constant.

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem. First we need a simple Lemma which we state as follows:

Lemma. For any real number $x>1$, we have the asymptotic

$$\sum_{n\leq x}\Omega(n)=x\ln\ln x+Ax+\mathrm{O}\left(\frac{x}{\ln x}\right),$$

where $A=\gamma+\sum\limits_{p}\left(\ln\left(1-\frac{1}{p}\right)+\frac{1}{p-1}\right)$, γ be the Euler constant.

Proof. See reference [4].

Now we use above Lemma to complete the proof of our theorem. For any real number $x\geqslant 2$, let M be a fixed positive integer such that

$$M^k\leqslant x<(M+1)^k.$$

Then from the definition of M we have the estimate

$$M=x^{\frac{1}{k}}+O(1). \tag{1}$$

For any prime p and positive integer α, note that $\Omega(p^\alpha)=\alpha p$ and

$$(x+1)^k=\sum_{i=0}^{k}C_k^i\cdot x^{k-i}.$$

Then from the definition of $b_k(n)$ and (1) we have

$$\begin{aligned}
&\sum_{n\leq x}\Omega(n+b_k(n)) = \sum_{1\leqslant t\leqslant M-1}\left(\sum_{t^k\leqslant n<(t+1)^k}\Omega(n+b_k(n))\right)+\sum_{M^k\leqslant n\leqslant x}\Omega(n+b_k(n))\\
&=\sum_{1\leqslant t\leqslant M-1}\left(\sum_{t^k\leqslant n<(t+1)^k}\Omega\left((t+1)^k\right)\right)+\mathrm{O}\left(\sum_{M^k\leqslant n\leqslant (M+1)^k}\Omega\left((M+1)^k\right)\right)\\
&=\sum_{1\leqslant t\leqslant M-1}k\left(C_k^1t^{k-1}+C_k^2t^{k-2}+\cdots+1\right)\Omega(t+1)+\mathrm{O}\left(x^{\frac{k-1}{k}+\varepsilon}\right)\\
&=k^2\sum_{1\leqslant t\leqslant M-1}(t+1)^{k-1}\Omega(t+1)+\mathrm{O}\left(x^{\frac{k-1}{k}+\varepsilon}\right)\\
&=k^2\sum_{1\leqslant t\leqslant M}t^{k-1}\Omega(t)+\mathrm{O}\left(x^{\frac{k-1}{k}+\varepsilon}\right),
\end{aligned}\tag{2}$$

where we have used the estimate $\Omega(n)\ll n^{\varepsilon}$.

Let $A(x)=\sum\limits_{n\leqslant x}\Omega(n)$, then by Able's identity (see Theorem 4.2 of reference [5]) and the above Lemma we can easily deduce that

$$\begin{aligned}
&\sum_{1\leqslant t\leqslant M}t^{k-1}\Omega(t)=M^{k-1}A(M)-\int_2^M A(t)\left(t^{k-1}\right)'dt+\mathrm{O}(1)\\
&=M^{k-1}\left(M\ln\ln M+AM+\mathrm{O}\left(\frac{M}{\ln M}\right)\right)\\
&-\int_2^M\left(t\ln\ln t+At+\mathrm{O}\left(\frac{t}{\ln t}\right)\right)(k-1)\,t^{k-2}dt+\mathrm{O}(1)\\
&=M^k\ln\ln M+AM^k+\mathrm{O}\left(\frac{M^k}{\ln M}\right)-\int_2^M\left((k-1)\,t^{k-1}\ln\ln t+(k-1)\,At^{k-1}\right)dt\\
&=M^k\ln\ln M+AM^k-\frac{k-1}{k}\left(M^k\ln\ln M+AM^k\right)+\mathrm{O}\left(\frac{M^k}{\ln M}\right)\\
&=\frac{1}{k}M^k\ln\ln M+\frac{1}{k}AM^k+\mathrm{O}\left(\frac{M^k}{\ln M}\right).
\end{aligned}\tag{3}$$

Note that

$$0\leqslant x-M^k<(M+1)^k-M^k=C_k^1M^{k-1}+C_k^2M^{k-2}+\cdots+1\ll x^{\frac{k-1}{k}}\tag{4}$$

and

$$\ln k+\ln\ln M\leqslant\ln\ln x<\ln k+\ln\ln(M+1)\leqslant\ln k+\ln\ln M+\mathrm{O}\left(x^{-\frac{1}{k}}\right).\tag{5}$$

From (3), (4) and (5) we have

$$\sum_{1\leqslant t\leqslant M}t^{k-1}\Omega(t)=\frac{1}{k}x\ln\ln x+\frac{1}{k}(A-\ln k)\,x+\mathrm{O}\left(\frac{x}{\ln x}\right).\tag{6}$$

Combining (2) and (6) we may immediately deduce the asymptotic formula

$$\sum_{n\leqslant x}\Omega(n+b_k(n))=kx\ln\ln x+k(A-\ln k)\,x+\mathrm{O}\left(\frac{x}{\ln x}\right).$$

This completes the proof of Theorem.

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The generalization of sequence of numbers with alternate common differences

Xiong Zhang, Yilin Zhang and Jianjie Ding

Shaanxi Institute of education, Xi'an 710061, China

Abstract In this paper, as generalizations of the number sequences with alternate common differences, two types of special sequence of numbers are discussed. One is the periodic sequences of numbers with two common differences; and the other is the periodic sequence of numbers with two common ratios. The formulus of the general term a_n and the sum of the first n term S_n are given respectively.
Keywords Sequence of numbers with alternate common differences, periodic number sequence with two common differences, periodic number sequence of numbers with two common ratios, general term a_n, the sum of the first n terms S_n.

§1. Introduction

In the paper[1], we have the definition like this: A sequence of numbers $\{a_n\}$ is called a sequence of numbers with alternate common differences if the following conditions are satisfied:

(i) $\forall k \in N, a_{2k} - a_{2k-1} = d_1$;

(ii) $\forall k \in N, a_{2k+1} - a_{2k} = d_2$ here $d_1(d_2)$ is called the first(the second)common difference of $\{a_n\}$.

We also give the formulas of the general term a_n and the sum of the first n terms S_n. In this paper, we'll discuss the generalization of sequence of numbers with alternate.

§2. Periodic number sequence two common differences

Definition 1.1. A sequence of numbers $\{a_n\}$ is called a periodic number sequence two common differences if the following conditions are satisfied:

(i) $\forall k = 0, 1, 2, \cdots a_{kt+1}, a_{kt+2}, a_{kt+3}, \cdots, a_{kt+t}$ is a finite arithmetic progression with d_1 as the common difference,where t is a constant natural numbers;

(ii) $\forall k = 0, 1, 2, \cdots a_{(k+1)t+1} = a_{(k+1)t} + d_2$.

We call the finite arithmetic progression "$a_{kt+1}, a_{kt+2}, a_{kt+3}, \cdots, a_{kt+t}$" the $(k+1)th$ period of $\{a_n\}$ and "$a_{(k+1)t}, a_{(k+1)t+1}$" the $(k+1)th$ interval of $\{a_n\}$; d_1 is named the common difference inside the periods and d_2 is called the interval common difference, t is called the number sequence $\{a_n\}^{'}s$ period.

In this section, $\{a_n\}$ denotes a periodic sequence of numbers with two common differences d_1 and d_2. It's easy to get that$\{a_n\}$ has the following form:

$$a_1, a_1+d_1, a_1+2d_1, \cdots, a_1+(t-1)d_1; a_1+(t-1)d_1+d_2, a_1+td_1+d_2,$$
$$a_1+(t+1)d_1+d_2, \cdots, a_1+(2t-2)d_1+d_2; a_1+(2t-2)d_1+2d_2, a_1+(2t-1)d_1+2d_2,$$
$$a_1+2td_1+2d_2, \cdots, a_1+(3t-3)d_1+d_2, \cdots. \quad (1)$$

Particularly, when $t=2, \{a_n\}$ becomes a sequence of numbers with alternate common differences d_1 and d_2; so the concept of a periodic number sequence with two common differences is a generalization of the concept of a number sequence with alternate common differences.

Theorem 1.1. The formula of the general term of (1) is

$$a_n = a_1 + (n-1-\left[\frac{n-1}{t}\right])d_1 + \left[\frac{n-1}{t}\right]d_2.$$

Proof.

$$\begin{aligned} a_n &= a_1+(n-1)d_1+(d_2-d_1)k \\ &= a_1+kd_2+[(n-1)-k]d_1 \\ &= a_1+\left[\frac{n-1}{t}\right]d_2+(n-1-\left[\frac{n-1}{t}\right])d_1. \end{aligned}$$

Here, k means the number of intervals, it can be proved easily that $k=\left[\frac{n-1}{t}\right]$.

Theorem 1.2. $\{a_n\}$ is a periodic number sequence with two common differences d_1 and d_2, the sum of the first n terms of $\{a_n\}S_n$ is:

$$S_n = n\,a_1+\frac{t(t-1)}{2}\left[\frac{n}{t}\right]^2 d_1+\frac{\left[\frac{n}{t}\right]\left(\left[\frac{n}{t}\right]-1\right)}{2}t\,d_2+$$
$$(\left[\frac{n}{t}\right](t-1)\,d_1+\left[\frac{n}{t}\right]d_2)(n-\left[\frac{n}{t}\right]t)+\frac{(n-\left[\frac{n}{t}\right]t)(n-\left[\frac{n}{t}\right]t-1)}{2}\,d_1\,.$$

Particularly, when $t|n$, suppose $\frac{n}{t}=k$, then

$$S_n = n\,a_1+\frac{t(t-1)}{2}\,k^2\,d_1+\frac{k(k-1)}{2}t\,d_2\,.$$

Proof. Let $M_{(k,t)}$ be the sum of the t terms of the $(k+1)th$ period.

Then

$$\begin{aligned} M_{(k,t)} &= t\,a_{(k-1)t+1}+\frac{t(t-1)}{2}\,d_1 \\ &= t[a_1+(k-1)\,d_2+((k-1)t-(k-1))\,d_1]+\frac{t(t-1)}{2}\,d_1 \\ &= t\,a_1+t(k-1)\,d_2+\frac{2k-1}{2}t(t-1)\,d_1\,. \end{aligned}$$

$$\begin{aligned} M_{(k+1,t)} &= t\,a_{kt+1}+\frac{t(t-1)}{2}\,d_1 \\ &= t[a_1+k\,d_2+(kt-k)\,d_1] \\ &= t\,a_1+tk\,d_2+\frac{2k+1}{2}t(t-1)\,d_1\,. \end{aligned}$$

Hence $M_{(k+1,t)}-M_{(k,t)}=t(t-1)d_1+td_2$. Therefore, the new sequence $\{M_{(k,t)}\}$ generated form $\{a_n\}$ is an arithmetic progression with $M_{(0,t)}=t\,a_1+\dfrac{t(t-1)}{2}\,d_1, d=t(t-1)d_1+td_2$. So the sum of the first $\left[\dfrac{n}{t}\right]t$ terms of $\{a_n\}$.

$$\begin{aligned} S_{\left[\frac{n}{t}\right]t} &= (t\,a_1+\frac{t(t-1)}{2}\,d_1)\left[\frac{n}{t}\right]+\frac{\left[\frac{n}{t}\right](\left[\frac{n}{t}\right]-1)}{2}(t(t-1)\,d_1+t\,d_2) \\ &= \left[\frac{n}{t}\right]t\,a_1+\frac{t(t-1)}{2}\left[\frac{n}{t}\right]^2 d_1+\frac{\left[\frac{n}{t}\right](\left[\frac{n}{t}\right]-1)}{2}t\,d_2\,. \end{aligned}$$

$$\begin{aligned} S_n-S_{\left[\frac{n}{t}\right]t} &= a_{\left[\frac{n}{t}\right]t+1}(n-\left[\frac{n}{t}\right])+\frac{(n-\left[\frac{n}{t}\right]t)(n-\left[\frac{n}{t}\right]t-1)}{2}\,d_1 \\ &= (a_1+\left[\frac{n}{t}\right](t-1)\,d_1+\left[\frac{n}{t}\right]d_2)(n-\left[\frac{n}{t}\right]t)+\frac{(n-\left[\frac{n}{t}\right]t)(n-\left[\frac{n}{t}\right]-1)}{2}\,d_1\,. \end{aligned}$$

Thus,

$$\begin{aligned} &S_n=S_{\left[\frac{n}{t}\right]t}+(S_n-S_{\left[\frac{n}{t}\right]t}) \\ &=n\,a_1+\frac{t(t-1)}{2}\left[\frac{n}{t}\right]^2 d_1+\frac{\left[\frac{n}{t}\right](\left[\frac{n}{t}\right]t-1)}{2}t\,d_2+(\left[\frac{n}{t}\right](t-1)\,d_1+\left[\frac{n}{t}\right]d_2)(n-\left[\frac{n}{t}\right]t) \\ &+\frac{(n-\left[\frac{n}{t}\right]t)(n-\left[\frac{n}{t}\right]t-1)}{2}\,d_1\,. \end{aligned}$$

Particularly, when $t|n$, suppose $\dfrac{n}{t}=k$, then

$$S_n=n\,a_1+\frac{t(t-1)}{2}\,k^2\,d_1+\frac{k(k-1)}{2}t\,d_2\,.$$

§3. Periodic number sequence with two common ratios

Definition 2.1. A sequence of numbers $\{a_n\}$ is called a periodic number sequence with two common ratios if the following conditions are satisfied:

(i) $\forall k=0,1,2,\cdots a_{kt+1},a_{kt+2},a_{kt+3},\cdots,a_{kt+t}$ is a finite geometric progression with q_1 as the common ratio, where t is a constant natural number;

(ii) $\forall k=0,1,2,\cdots a_{(k+1)t+1}=a_{(k+1)t}q_2$, where q_2 is a constant natural number;

We call the finite geometric progression "$a_{kt+1},a_{kt+2},a_{kt+3},\cdots,a_{kt+t}$" the $(k+1)th$ period of $\{a_n\}$ and "$a_{(k+1)t},a_{(k+1)t+1}$" the $(k+1)th$ interval of $\{a_n\}$; q_1 is named the common ratio inside the periods and q_2 is called the interval common ratio, t is called the number sequence $\{a_n\}'s$ period.

In this section, $\{a_n\}$ denotes a periodic sequence of numbers with two common ratios q_1 and q_2. It's easy to get that $\{a_n\}$ has the following form:

$$\begin{aligned} a_1, a_1q_1, a_1q_1^2, \cdots, a_1q_1^{t-1}; a_1q_2q_1^{t-1}, a_1q_2q_1^t, a_1q_2q_1^{t+1}, \cdots, a_1q_2q_1^{2t-2}; \\ a_1q_2^2q_1^{2t-2}, a_1q_2^2q_1^{2t-1}, a_1q_2^2q_1^{2t}, \cdots, a_1q_2^2q_1^{3t-3}, \cdots. \end{aligned} \tag{2}$$

The formula of the general term of (2) is

$$a_n = a_1 q_1^{(n-1-[\frac{n-1}{t}])} q_2^{[\frac{n-1}{t}]}.$$

Let $M_{(k,t)}$ be the sum of the t terms of the $(k+1)th$ period. Therefore, the new sequence $\{M_{(k,t)}\}$ generated form $\{a_n\}$ is an geometric progression with $M_{(0,t)} = \frac{a_1(1-q_1^{'})}{1-q_1}, q = q_2q_1^{t-1}$. So the sum of the first n terms of $\{a_n\}$.

Theorem 2.2. $\{a_n\}$ is a periodic number sequence with two common ratios q_1 and q_2, the sum of the first n terms of $\{a_n\}S_n$ is

$$S_n = \frac{\frac{a_1(1-{q_1}^t)}{1-q_1}(1-(q_2\,{q_1}^{t-1})^{\left[\frac{n}{t}\right]})}{1-q_2\,{q_1}^{t-1}} + \frac{a_1\,{q_2}^{\left[\frac{n}{t}\right]}\,{q_1}^{\left[\frac{n}{t}\right](t-1)}(1-{q_1}^{n-\left[\frac{n}{t}\right]t})}{1-q_1}.$$

Particularly, when $t|n$, suppose$\dfrac{n}{t} = k$, then

$$S_n = \frac{a_1(1-{q_1}^t)(1-(q_2\,{q_1}^{t-1})^k)}{(1-q_1)(1-q_2\,{q_1}^{t-1})}.$$

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An equation related to the Smarandache function and its positive integer solutions

Wenjing Xiong

Department of Mathematics and Physics, Shaanxi Institute of Education
Xi'an, 710061, Shaanxi, P.R.China

Abstract For any positive integer n, the famous F. Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : \ n \mid m!, \ n \in N\}$. The Smarandache reciprocal function $S_c(n)$ is defined as $S_c(n) = \max\{m : \ y \mid n! \text{ for all } 1 \leq y \leq m, \text{ and } \ m+1 \dagger n!\}$. That is, $S_c(n)$ is the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$. The main purpose of this paper is using the elementary method to study the solvability of an equation involving the Smarandache function $S(n)$ and the Smarandache reciprocal function $S_c(n)$, and obtain its all positive integer solutions.

Keywords The Smarandache function, the Smarandache reciprocal function, equation, positive integer solutions.

§1. Introduction and result

For any positive integer n, the famous F. Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : \ n \mid m!, \ n \in N\}$. It is easy to find that the first few values of this function are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $S(11) = 11$, $S(12) = 4$, $\cdots$. About the elementary properties of $S(n)$, many authors had studied it, and obtained some interesting results, see references [1]-[5]. For example, Xu Zhefeng [5] studied the value distribution problem of $S(n)$, and proved the following conclusion:

Let $P(n)$ denote the largest prime factor of n, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} (S(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ denotes the Riemann zeta-function.

On the other hand, in reference [6], A. Murthy introduced another function $S_c(n)$, which called the Smarandache reciprocal function. It is defined as the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$. That is, $S_c(n) = \max\{m : \ y \mid n! \text{ for all } 1 \leq y \leq$

$m,$ and $m+1 \dagger n!\}$. For example, the first few values of $S_c(n)$ are:

$$S_c(1)=1,\ S_c(2)=2,\ S_c(3)=3,\ S_c(4)=4,\ S_c(5)=6,\ S_c(6)=6,\ S_c(7)=10,$$
$$S_c(8)=10,\ S_c(9)=10,\ S_c(10)=10,\ S_c(11)=12,\ S_c(12)=12,\ S_c(13)=16,$$
$$S_c(14)=16,\ S_5(15)=16,\ S_c(16)=16,\ S_c(17)=18,\ S_c(18)=18,\ \cdots.$$

A. Murthy [6] studied the elementary properties of $S_c(n)$, and proved the following conclusion:

If $S_c(n)=x$ and $n\neq 3$, then $x+1$ is the smallest prime greater than n.

The main purpose of this paper is using the elementary method to study the solvability of an equation involving the Smarandache function $S(n)$ and the Smarandache reciprocal function $S_c(n)$, and obtain its all positive integer solutions. That is, we shall prove the following:

Theorem. For any positive integer n, the equation

$$S_c(n)+S(n)=n$$

holds if and only if $n=1$, 2, 3 and 4.

§2. Proof of the theorem

In this section, we shall prove our Theorem directly. First we need an estimate for $\pi(x)$, the number of all primes $\leq x$. From J. B. Rosser and L. Schoenfeld [7] we have the estimate

$$\pi(x)<\frac{x}{\ln x}\left(1+\frac{3}{2\cdot\ln x}\right)\quad \text{for}\quad x>1$$

and

$$\pi(x)>\frac{x}{\ln x}\left(1+\frac{1}{2\cdot\ln x}\right)\quad \text{for}\quad x>59.$$

Using these estimates and some calculating we can prove that there must exist a prime between n and $\frac{3}{2}n$, if $n\geq 59$. So from this conclusion and A. Murthy [6] we have the estimate

$$S_c(n)<\frac{3}{2}\cdot n,\quad \text{if}\quad n\geq 59. \tag{1}$$

If $1\leq n\leq 59$, it is easy to check that $n=1$, 2, 3 and 4 satisfy the equation $S_c(n)+S(n)=n$. Now we can prove that n does not satisfy the equation $S_c(n)+S(n)=n$, if $n>59$. In fact this time, if n be a prime $p>59$, then from reference [6] we know that $S_c(p)>p$ and $S(p)=p$, so $S_c(p)+S(p)>2p$. If n has more than two prime divisors, from the properties of Smarandache function $S(n)$ we know that

$$S(n)=\max_{1\leq i\leq r}\{S(p_i^{\alpha_i})\}\leq\max_{1\leq i\leq r}\{\alpha_i\cdot p_i\},$$

if $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the factorization of n into prime powers. From this formula we may immediately deduce the estimate

$$S(n)\leq\frac{1}{2}\cdot n. \tag{2}$$

Combining (1) and (2) we can deduce that if $n > 59$ has more than two prime divisors, then

$$S_c(n) + S(n) < \frac{3}{2} \cdot n + \frac{1}{2} \cdot n = 2n.$$

So n does not satisfy the equation $S_c(n) + S(n) = 2n$.

If $n = p^\alpha > 59$ be a power of prime p, and $\alpha \geq 2$, then note that $S(2^\alpha) \leq \frac{1}{2} \cdot 2^\alpha$, $\alpha \geq 3$; $S(3^\alpha) \leq \frac{1}{2} \cdot 3^\alpha$, $\alpha \geq 3$; $S(p^\alpha) \leq \frac{1}{2} \cdot p^\alpha$, $\alpha \geq 2$, $p \geq 5$. We also have $S(n) \leq \frac{1}{2} \cdot n$, and therefore,

$$S_c(p^\alpha) + S(p^\alpha) < \frac{3}{2} \cdot p^\alpha + \frac{1}{2} \cdot p^\alpha = 2 \cdot p^\alpha.$$

So the equation $S_c(n) + S(n) = 2n$ has no positive integer solution if $n > 59$. This completes the proof of Theorem.

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On right C-rpp semigroups

Huaiyu Zhou

School of Statistics, Xi'an University of Finance and Economics,
Xi'an 710061, China

E-mail: zhouhy9526@126.com

Abstract An rpp semigroup S is called a right $C-rpp$ semigroup if $\mathcal{L}^* \vee \mathcal{R}$ is a congruence on S and $Se \subset eS$ for all $e \in E(S)$. This paper studies some properties on right $C-rpp$ semigroups by using the concept of right Δ-product. And, we obtained that the right $C-rpp$ semigroups whose set of idempotents forms a right normal band are a strong semilattice of direct products of left cancellative monoids and right zero bands.

Keywords Right Δ-product, right normal band, left cancellative monoids.

§1. Introduction

A semigroup S is called an rpp semigroup if all its principal right ideals $aS^1(a \in S)$, regard as right S^1-systems, are projective. This class of semigroups and its subclasses have been extensively studied by J.B.Fountain and other authors (see [1-7]). On a semigroup S, the Green's star relation $\mathcal{L}^*$ is defined by $(a,b) \in \mathcal{L}^*$ if and only if the elements a, b of S are related by the usual Green's relation $\mathcal{L}$ on some oversemigroup of S. It was then shown by J.B.Fountain [2] that a monoid S is rpp if and only if every $\mathcal{L}^*$-class contains an idempotent. Thus, a semigroup S is rpp if and only if every $\mathcal{L}^*$-class of S contains at least one idempotent. Dually, we can define lpp semigroups and a semigroup which is both rpp and lpp is called abundant[3]. Abundant semigroups and rpp semigroups are generalized regular semigroups.

It is noted that rpp semigroups with central idempotents have similar structure as Clifford semigroups. This kind of rpp semigroups was called the $C-rpp$ semigroups by J.B.Fonutain. He has proved that a $C-rpp$ semigroup can be described as a strong semilattice of left cancellative monoids.

In order to generalize the above result of Fountain, Y.Q.Guo, K.P.Shum and P.Y.Zhu have introduced the concept of strongly rpp semigroups in [4]. They considered an rpp semigroup S with a set of idempotents $E(S)$. For $\forall a \in S$, let the enevelope of a be $\mathcal{M}_a = \{e \in E(S) | S^1a \subset S^1e$ and $\forall x, y \in S^1, ax = ay \Rightarrow ex = ey\}$. Surely, $\mathcal{M}_a$ consists of the idempotents in the $\mathcal{L}^*$-class of a. Then the authors in [4] called the semigroup S strongly rpp if there exists a unique e in $\mathcal{M}_a$ such that $ea = a$ for $\forall a \in S$. Now, we call a semigroup S a left $C-rpp$ semigroup[4] if S is strongly rpp and $\mathcal{L}^*$ is a semilattice congruence on S. Y.Q.Guo called an rpp semigroup S a right $C-rpp$ semigroup[5] if $\mathcal{L}^* \vee \mathcal{R}$ is a congruence on S and $Se \subset eS$ for $\forall e \in E(S)$. He has shown that a right $C-rpp$ semigroup S can be expressed as a semilattice Y of direct

products M_a and B_a, where M_a is a left cancellative monoid and B_a is a right zero band for $\forall\alpha \in Y$.

In this paper, we will give some properties on right $C-rpp$ semigroups by the result of K.P.Shum and X.M.Ren in [8]. Then, by using the concept of right Δ-product, we study the right $C-rpp$ semigroups whose set of idempotents forms a right normal band. We will see that this kind of semigroups is a strong semilattice of direct products of left cancellative monoids and right zero bands.

Terminologies and notations which are not mentioned in this paper should be referred to [8] and also the text of J.M.Howie [9].

§2. Preliminaries

In this section, we simply introduce the concept of right Δ-product of semigroups and the structure of right $C-rpp$ semigroups. These are introduced by K.P.Shum and X.M.Ren in [8].

We let Y be a semilattice and $M = [Y; M_\alpha, \theta_{\alpha,\beta}]$ is a strong semilattice of cancellative monoids M_α with structure homomorphism $\theta_{\alpha,\beta}$. Let $\Lambda = \cup_{\alpha\in Y}\Lambda_\alpha$ be a semilattice decomposition of right regular band Λ into right zero band Λ_α. For $\forall\alpha \in Y$, we form the Cartesian product $S_\alpha = M_\alpha \times \Lambda_\alpha$.

Now, for $\forall\alpha, \beta \in Y$ with $\alpha \geqslant \beta$ and the right transformation semigroup $\mathcal{J}^*(*_\beta)$, we define a mapping

$$\Phi_{\alpha,\beta} : S_\alpha \to \mathcal{J}^*(*_\beta)$$

by $u \mapsto \varphi^u_{\alpha,\beta}$ satisfying the following conditions:

(P1): If $(a, i) \in S_\alpha$, $i' \in \Lambda_\alpha$, then $i'\varphi^{(a,i)}_{\alpha,\alpha} = i$;

(P2): For $\forall(a, i) \in S_\alpha, (b, j) \in S_\beta$, we consider the following situation separately:

(a) $\varphi^{(a,i)}_{\alpha,\alpha\beta}\varphi^{(b,j)}_{\beta,\alpha\beta}$ is a constant mapping on $\Lambda_{\alpha\beta}$ and we denote the constant value by $\langle\varphi^{(a,i)}_{\alpha,\alpha\beta}\varphi^{(b,j)}_{\beta,\alpha\beta}\rangle$;

(b) If $\alpha, \beta, \delta \in Y$ with $\alpha\beta \geqslant \delta$ and $\langle\varphi^{(a,i)}_{\alpha,\alpha\beta}\varphi^{(b,j)}_{\beta,\alpha\beta}\rangle = k$, then $\varphi^{(ab,k)}_{\alpha\beta,\delta} = \varphi^{(a,i)}_{\alpha,\delta}\varphi^{(b,j)}_{\beta,\delta}$;

(c) If $\varphi^{(\omega,\lambda)}_{\gamma,\gamma\alpha}\varphi^{(a,i)}_{\alpha,\gamma\alpha} = \varphi^{(\omega,\lambda)}_{\gamma,\gamma\beta}\varphi^{(b,j)}_{\beta,\gamma\beta}$ for $\forall(\omega, \lambda) \in S_\gamma$, then $\varphi^{(1_\gamma,\lambda)}_{\gamma,\gamma\alpha}\varphi^{(a,i)}_{\alpha,\gamma\alpha} = \varphi^{(1_\gamma,\lambda)}_{\gamma,\gamma\beta}\varphi^{(b,j)}_{\beta,\gamma\beta}$, where 1_γ is the identity of the monoid M_γ.

We now form the set union $S = \bigcup_{\alpha\in Y} S_\alpha$ and define a multiplication " $\circ$ " on S by

$$(a, i) \circ (b, j) = (ab, \langle\varphi^{(a,i)}_{\alpha,\alpha\beta}\varphi^{(b,j)}_{\beta,\alpha\beta}\rangle) \qquad (*)$$

After straightforward verification, we can verify that the multiplication" $\circ$ "satisfies the associative law and hence $(S, \circ)$ becomes a semigroup. We call the above constructed semigroup the right Δ-product of semigroup M and Λ on Y, under the structure mapping $\Phi_{\alpha,\beta}$. We denote this semigroup $(S, \circ)$ by $S = M\Delta^*_{Y,\Phi}\Lambda$.

Lemma 2.1. (See [8] Theorem 1.1). Let $M = [Y; M_\alpha, \theta_{\alpha,\beta}]$ be a strong semilattice of cancellative monoids M_α with structure homomorphism $\theta_{\alpha,\beta}$. Let $\Lambda = \bigcup_{\alpha\in Y}\Lambda_\alpha$ be a semilattice decomposition of right regular band Λ into right zero band Λ_α on the semilattice Y. Then the right Δ-product of M and Λ, denoted by $M\Delta^*_{Y,\Phi}\Lambda$, is a right $C-rpp$ semigroup. Conversely, every right $C-rpp$ semigroup can be constructed by using this method.

§3. Some properties and main result

In this section, we will first give some properties, by using right Δ-product of semigroups, for right $C-rpp$ semigroups which have been stated in the introduction. Then, we will obtain the structure of right $C-rpp$ semigroups whose set of idempotents forms a right normal band.

Theorem 3.1. Let S be a right $C-rpp$ semigroup. Then the following statements hold:

(1) For $\forall u \in RegS$, $Su \subseteq uS$;

(2) For $\forall e \in E(S)$, the mapping η_e: $x \mapsto xe(\forall x \in S^1)$ is a semigroup homomorphism from S^1 onto S^1e.

Proof. (1) We first assume that $S = M\Delta^*_{Y,\Phi}\Lambda$ is an arbitrary right $C-rpp$ semigroup. For $\forall u = (a,i) \in S_\alpha \cap RegS$, there exists $x = (b,j) \in S_\beta$ such that $uxu = u$ and $xux = x$. we can easily know $\alpha = \beta$ by the multiplication of semigroups. Hence for $\forall x \in S_\alpha$, from $(*)$ and (P1) we have $(b,j) = x = xux = (bab,j)$. So $bab = b = b1_\beta = b1_\alpha$, where 1_α is the identity element of M_α. By the left cancelltivity of M_α, we immediately obtain $ab = 1_\alpha$.

For $\gamma \in Y$ and $v = (c,k) \in S_\gamma$, let $w = ((b\theta_{\alpha,\gamma\alpha})(ca), \langle \varphi^v_{\gamma,\gamma\alpha}\varphi^u_{\alpha,\gamma\alpha}\rangle) \in S_{\gamma\alpha}$, where $\theta_{\alpha,\gamma\alpha}$ is a semigroup homomorphism from M_α onto $M_{\gamma\alpha}$, and $M_{\gamma\alpha}$ is a left cancellative monoid. This leads to $1_\alpha\theta_{\alpha,\gamma\alpha} = 1_{\gamma\alpha}$. Hence,

$$\begin{aligned}
uw &= (a,i)((b\theta_{\alpha,\gamma\alpha})(ca), \langle \varphi^v_{\gamma,\gamma\alpha}\varphi^u_{\alpha,\gamma\alpha}\rangle)\\
&= ((a\theta_{\alpha,\gamma\alpha})\theta_{\gamma\alpha,\gamma\alpha}(b\theta_{\alpha,\gamma\alpha})(ca), \langle \varphi^u_{\alpha,\gamma\alpha}\varphi^w_{\gamma\alpha,\gamma\alpha}\rangle)\\
&= ((a\theta_{\alpha,\gamma\alpha})(b\theta_{\alpha,\gamma\alpha})(ca), \langle \varphi^v_{\gamma,\gamma\alpha}\varphi^u_{\alpha,\gamma\alpha}\rangle)\\
&= ((ab)\theta_{\alpha,\gamma\alpha}(ca), \langle \varphi^v_{\gamma,\gamma\alpha}\varphi^u_{\alpha,\gamma\alpha}\rangle)\\
&= (1_\alpha\theta_{\alpha,\gamma\alpha}(ca), \langle \varphi^v_{\gamma,\gamma\alpha}\varphi^u_{\alpha,\gamma\alpha}\rangle)\\
&= (1_{\gamma\alpha}(ca), \langle \varphi^v_{\gamma,\gamma\alpha}\varphi^u_{\alpha,\gamma\alpha}\rangle)\\
&= (ca, \langle \varphi^v_{\gamma,\gamma\alpha}\varphi^u_{\alpha,\gamma\alpha}\rangle)\\
&= (c,k)(a,i)\\
&= vu.
\end{aligned}$$

This shows $Su \subseteq uS$.

(2)For $\forall e \in E(S)$ and $\forall x, y \in S^1$, if $y = 1$, then we have immediately

$$\eta_e(x \cdot 1) = \eta_e(x) = xe = xee = \eta_e(x)\eta_e(1).$$

If $x \in S$, then we know there exists $z \in S$ such that $ye = ez$ by using $(*)$ and $e \in RegS$. Hence,

$$\eta_e(xy) = (xy)e = x(ye) = x(ez) = xe(ez) = (xe)(ye) = \eta_e(x)\eta_e(y).$$

This shows that η_e is a semigroup homomorphism.

In the following section, we proceed to study the structure of right $C-rpp$ semigroups whose set of idempotents forms a right normal band.

Definition 3.2. A band E is called a right normal band if $\forall e, f, g \in E$ such that $efg = feg$.

Theorem 3.3. Let $S = M\Delta^*_{Y,\Phi}\Lambda$ be a right $C-rpp$ semigroup. $E(S)$ is the set of idempotents of S. Then the following statements are equivalent:

(1) S is a strong semilattice of $M_\alpha \times \Lambda_\alpha$;

(2) $E(S)$ is a right normal band.

Where $M = [Y; M_\alpha, \theta_{\alpha,\beta}]$ be a strong semilattice of left cancellative monoids M_α with structure homomorphism $\theta_{\alpha,\beta}$, $\Lambda = \cup_{\alpha\in Y}\Lambda_\alpha$ be a semilattice decomposition of right regular band Λ into right zero band Λ_α.

Proof. (1)⇒(2). Let S be strong semilattice Y of $S_\alpha = M_\alpha \times \Lambda_\alpha$ with structure homomorphism $\Psi_{\alpha,\beta}$ for $\alpha, \beta \in Y$ and $\alpha \geqslant \beta$. From [8] we knew $E(S) = \bigcup_{\alpha\in Y}\{(1_\alpha, i) \in M_\alpha \times \Lambda_\alpha | i \in \Lambda_\alpha\}$, where 1_α is the identity element of M_α. If $(1_\alpha, i) \in S_\alpha, (1_\beta, j) \in S_\beta$, then $(1_\alpha, i)\psi_{\alpha,\alpha\beta}$ is an element in $E_{\alpha\beta}$. Since $E_{\alpha\beta}$ is a right zero band, we have

$$(1_\alpha, i)(1_\beta, j) = (1_\alpha, i)\psi_{\alpha,\alpha\beta}(1_\beta, j)\psi_{\beta,\alpha\beta} = (1_\beta, j)\psi_{\beta,\alpha\beta}.$$

Consequently, for any idempotents $(1_\alpha, i), (1_\beta, j)$ and $(1_\gamma, k)$, we have

$$(1_\alpha, i)(1_\beta, j)(1_\gamma, k) = (1_\alpha, i)\psi_{\alpha,\alpha\beta\gamma}(1_\beta, j)\psi_{\beta,\alpha\beta\gamma}(1_\gamma, k)\psi_{\gamma,\alpha\beta\gamma} = (1_\gamma, k)\psi_{\gamma,\alpha\beta\gamma}.$$
$$(1_\beta, j)(1_\alpha, i)(1_\gamma, k) = (1_\beta, j)\psi_{\beta,\alpha\beta\gamma}(1_\alpha, i)\psi_{\alpha,\alpha\beta\gamma}(1_\gamma, k)\psi_{\gamma,\alpha\beta\gamma} = (1_\gamma, k)\psi_{\gamma,\alpha\beta\gamma}.$$

Thus,

$$(1_\alpha, i)(1_\beta, j)(1_\gamma, k) = (1_\beta, j)(1_\alpha, i)(1_\gamma, k).$$

This shows that $E(S)$ is a right normal band.

(2)⇒(1). If $E(S)$ is a right normal band, then we knew that $E(S)$ is a strong semilattice of right zero band, and every right zero band is just a $\mathcal{J}(= \mathcal{D})$-class of $E(S)$. As $E(S)$ itself is a semilattice of right zero bands $E_\alpha = \{(1_\alpha, i) \mid i \in \Lambda_\alpha\}$, each E_α is just a $\mathcal{J}$-class of $E(S)$. This means that $E(S)$ is a strong semilattice of E_α. Let the strong semilattice structure homomorphism be $\xi_{\alpha,\beta}$, where $\alpha, \beta \in Y$ and $\alpha \geqslant \beta$. Then for any idempotents $(1_\alpha, i), (1_\beta, j)$, we have

$$(1_\alpha, i)(1_\beta, j) = (1_\alpha, i)\xi_{\alpha,\alpha\beta}(1_\beta, j)\xi_{\beta,\alpha\beta} = (1_\beta, j)\xi_{\beta,\alpha\beta}.$$

Let $\theta_{\alpha,\beta}$ be the strong semilattice structure homomorphism of the $C-rpp$ compotent $M_s = \bigcup_{\alpha\in Y} M_\alpha$ of S. By virtue of the right normality of $E(S)$, for $\forall(a, i) \in S_\alpha$ and $j_1, j_2 \in \Lambda_\beta \bigcup \Lambda_{\alpha\beta}$, we have

$$\begin{aligned}
(a\theta_{\alpha,\alpha\beta}, \langle\varphi^{(a,i)}_{\alpha,\alpha\beta}\varphi^{(1_\beta,j_1)}_{\beta,\alpha\beta}\rangle) &= (a, i)(1_\beta, j_1)\\
&= (1_\alpha, i)(a, i)(1_\beta, j_1)\\
&= (1_\alpha, i)(a\theta_{\alpha,\alpha\beta}, \langle\varphi^{(a,i)}_{\alpha,\alpha\beta}\varphi^{(1_\beta,j_1)}_{\beta,\alpha\beta}\rangle)\\
&= (1_\alpha, i)(a, i)(1_{\alpha\beta}, \langle\varphi^{(a,i)}_{\alpha,\alpha\beta}\varphi^{(1_\beta,j_1)}_{\beta,\alpha\beta}\rangle)\\
&= (a, i)(1_\alpha, i)(1_{\alpha\beta}, \langle\varphi^{(a,i)}_{\alpha,\alpha\beta}\varphi^{(1_\beta,j_1)}_{\beta,\alpha\beta}\rangle)\\
&= (a, i)(1_\alpha, i)\xi_{\alpha,\alpha\beta}(1_{\alpha\beta}, \langle\varphi^{(a,i)}_{\alpha,\alpha\beta}\varphi^{(1_\beta,j_1)}_{\beta,\alpha\beta}\rangle)\xi_{\alpha\beta,\alpha\beta}\\
&= (a, i)(1_\alpha, i)(1_{\alpha\beta}, j_2)\\
&= (a, i)(1_{\alpha\beta}, j_2)\\
&= (a, i)(1_\beta, j_2)\\
&= (a\theta_{\alpha,\alpha\beta}, \langle\varphi^{(a,i)}_{\alpha,\alpha\beta}\varphi^{(1_\beta,j_2)}_{\beta,\alpha\beta}\rangle).
\end{aligned}$$

This shows $\langle\varphi^{(a,i)}_{\alpha,\alpha\beta}\varphi^{(1_\beta,j_1)}_{\beta,\alpha\beta}\rangle = \langle\varphi^{(a,i)}_{\alpha,\alpha\beta}\varphi^{(1_\beta,j_2)}_{\beta,\alpha\beta}\rangle$.

Now, for $\forall \alpha, \beta \in Y, \alpha \geqslant \beta, i \in \Lambda_\alpha, j_0 \in \Lambda_\beta$, we define a mapping $\Psi_{\alpha,\beta}$: $S_\alpha \to S_\beta$ by $(a,i) \mapsto (a\theta_{\alpha,\alpha\beta}, \langle \varphi_{\alpha,\alpha\beta}^{(a,i)} \varphi_{\beta,\alpha\beta}^{(1_\beta,j_0)} \rangle)$.

By the same arguments as the previous one, we can get the above mapping $\Psi_{\alpha,\beta}$ is independent of the choices $j_0 \in \Lambda_\beta$.

Clearly, $\Psi_{\alpha,\alpha}$ is an identity mapping for $\forall \alpha \in Y$.

Furthermore, for $\forall (a,i), (b,j) \in S_\alpha, \alpha \geqslant \beta$ and $j_0 \in \Lambda_\beta$, we have

$$\begin{aligned}(a,i)(b,j)\Psi_{\alpha,\beta} &= (ab,j)\Psi_{\alpha,\beta} \\ &= ((ab)\theta_{\alpha,\alpha\beta}, \langle \varphi_{\alpha,\alpha\beta}^{(ab,j)} \varphi_{\beta,\alpha\beta}^{(1_\beta,j_0)} \rangle) \\ &= (a\theta_{\alpha,\alpha\beta} b\theta_{\alpha,\alpha\beta}, \langle \varphi_{\alpha,\alpha\beta}^{(a,i)} \varphi_{\alpha,\alpha\beta}^{(b,j)} \varphi_{\beta,\alpha\beta}^{(1_\beta,j_0)} \rangle) \\ &= (a\theta_{\alpha,\alpha\beta}, \langle \varphi_{\alpha,\alpha\beta}^{(a,i)} \varphi_{\beta,\alpha\beta}^{(1_\beta,j_0)} \rangle)(b\theta_{\alpha,\alpha\beta}, \langle \varphi_{\alpha,\alpha\beta}^{(b,j)} \varphi_{\beta,\alpha\beta}^{(1_\beta,j_0)} \rangle) \\ &= (a,i)\Psi_{\alpha,\beta}(b,j)\Psi_{\alpha,\beta}.\end{aligned}$$

Thus, $\Psi_{\alpha,\beta}$ is indeed a homomorphism from S_α onto S_β.

For $\forall \alpha, \beta, \gamma \in Y$ satisfying $\alpha \geqslant \beta \geqslant \gamma$, and $j_0 \in \Lambda_\beta, k_0 \in \Lambda_\gamma$, we have

$$\begin{aligned}(a,i)\Psi_{\alpha,\beta}\Psi_{\beta,\gamma} &= (a\theta_{\alpha,\alpha\beta}, \langle \varphi_{\alpha,\alpha\beta}^{(a,i)} \varphi_{\beta,\alpha\beta}^{(1_\beta,j_0)} \rangle)\Psi_{\beta,\gamma} \\ &= (a\theta_{\alpha,\beta}, \langle \varphi_{\alpha,\beta}^{(a,i)} \varphi_{\beta,\beta}^{(1_\beta,j_0)} \rangle)\Psi_{\beta,\gamma} \\ &= (a\theta_{\alpha,\beta}, j_0)\Psi_{\beta,\gamma} \\ &= (a\theta_{\alpha,\beta}\theta_{\beta,\gamma}, \langle \varphi_{\beta,\gamma}^{(a\theta_{\alpha,\beta},j_0)} \varphi_{\gamma,\gamma}^{(1_\beta,k_0)} \rangle) \\ &= (a\theta_{\alpha,\gamma}, k_0) \\ &= (a\theta_{\alpha,\gamma}, \langle \varphi_{\alpha,\gamma}^{(a,i)} \varphi_{\gamma,\gamma}^{(1_\beta,k_0)} \rangle) \\ &= (a,i)\Psi_{\alpha,\gamma}.\end{aligned}$$

In other words, we have $\Psi_{\alpha,\beta}\Psi_{\beta,\gamma} = \Psi_{\alpha,\gamma}$. Summing up all the above discussion, we are now ready to construct a strong semilattice $\overline{S}$ of S_α with the above structure homomorphism $\Psi_{\alpha,\beta}$. Clearly, $\overline{S} = S$ as sets. The remaining part is to show that $\overline{S} = S$ as semigroup as well.

Denote the multiplication in $\overline{S}$ by $*$, Then for $\forall (a,i) \in S_\alpha, (b,j) \in S_\beta$ and $j_0 \in \Lambda_{\alpha\beta}$, we have

$$\begin{aligned}(a,i) * (b,j) &= (a,i)\Psi_{\alpha,\alpha\beta}(b,j)\Psi_{\beta,\alpha\beta} \\ &= (a\theta_{\alpha,\alpha\beta}, \langle \varphi_{\alpha,\alpha\beta}^{(a,i)} \varphi_{\beta,\alpha\beta}^{(1_\beta,j_0)} \rangle)(b\theta_{\beta,\alpha\beta}, \langle \varphi_{\beta,\alpha\beta}^{(b,j)} \varphi_{\beta,\alpha\beta}^{(1_\beta,j_0)} \rangle) \\ &= (a,i)(b,j).\end{aligned}$$

Thus, it can be seen that the multiplication $*$ of $\overline{S}$ is exactly the same as the usual semigroup multiplication of S. This shows that S is a strong semilattice of $M_\alpha \times \Lambda_\alpha$.

The proof is completed.

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On the pseudo Smarandache square-free function

Bin Cheng

Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China

Abstract For any positive integer n, the famous Pseudo Smarandache Square-free function $Z_w(n)$ is defined as the smallest positive integer m such that m^n is divisible by n. That is, $Z_w(n) = \min\{m : n|m^n, m \in N\}$, where N denotes the set of all positive integers. The main purpose of this paper is using the elementary method to study the properties of $Z_w(n)$, and give an inequality for it. At the same time, we also study the solvability of an equation involving the Pseudo Smarandache Square-free function, and prove that it has infinity positive integer solutions.

Keywords The Pseudo Smarandache Square-free function, Vinogradov's three-primes theorem, inequality, equation, positive integer solution.

§1. Introduction and results

For any positive integer n, the famous Pseudo Smarandache Square-free function $Z_w(n)$ is defined as the smallest positive integer m such that m^n is divisible by n. That is,

$$Z_w(n) = \min\{m : \ n|m^n, \ m \in N\},$$

where N denotes the set of all positive integers. This function was proposed by Professor F. Smarandache in reference [1], where he asked us to study the properties of $Z_w(n)$. From the definition of $Z_w(n)$ we can easily get the following conclusions: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers, then $Z_w(n) = p_1 p_2 \cdots p_r$. From this we can get the first few values of $Z_w(n)$ are: $Z_w(1) = 1$, $Z_w(2) = 2$, $Z_w(3) = 3$, $Z_w(4) = 2$, $Z_w(5) = 5$, $Z_w(6) = 6$, $Z_w(7) = 7$, $Z_w(8) = 2$, $Z_w(9) = 3$, $Z_w(10) = 10$, $\cdots$. About the elementary properties of $Z_w(n)$, some authors had studied it, and obtained some interesting results, see references [2], [3] and [4]. For example, Maohua Le [3] proved that

$$\sum_{n=1}^{\infty} \frac{1}{(Z_w(n))^{\alpha}}, \ \alpha \epsilon R, \ \alpha > 0$$

is divergence. Huaning Liu [4] proved that for any real numbers $\alpha > 0$ and $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} (Z_w(n))^{\alpha} = \frac{\zeta(\alpha+1) x^{\alpha+1}}{\zeta(2)(\alpha+1)} \prod_{p} \left[1 - \frac{1}{p^{\alpha}(p+1)}\right] + O\left(x^{\alpha+\frac{1}{2}+\epsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Now, for any positive integer $k > 1$, we consider the relationship between $Z_w\left(\prod_{i=1}^{k} m_i\right)$ and $\sum_{i=1}^{k} Z_w(m_i)$. In reference [2], Felice Russo suggested us to study the relationship between them. For this problem, it seems that none had studied it yet, at least we have not seen such a paper before. The main purpose of this paper is using the elementary method to study this problem, and obtained some progress on it. That is, we shall prove the following:

Theorem 1. Let $k > 1$ be an integer, then for any positive integers m_1, m_2, $\cdots$, m_k, we have the inequality

$$\sqrt[k]{Z_w\left(\prod_{i=1}^{k} m_i\right)} < \frac{\sum_{i=1}^{k} Z_w(m_i)}{k} \leq Z_w\left(\prod_{i=1}^{k} m_i\right),$$

and the equality holds if and only if all $m_1,\ m_2,\ \cdots,\ m_k$ have the same prime divisors.

Theorem 2. For any positive integer $k \geq 1$, the equation

$$\sum_{i=1}^{k} Z_w(m_i) = Z_w\left(\sum_{i=1}^{k} m_i\right)$$

has infinity positive integer solutions $(m_1,\ m_2,\ \cdots,\ m_k)$.

§2. Proof of the theorems

In this section, we shall prove our Theorems directly. First we prove Theorem 1. For any positive integer $k > 1$, we consider the problem in two cases:

(a). If $(m_i,\ m_j) = 1,\ i,\ j = 1,\ 2,\ \cdots,\ k$, and $i \neq j$, then from the multiplicative properties of $Z_w(n)$, we have

$$Z_w\left(\prod_{i=1}^{k} m_i\right) = \prod_{i=1}^{k} Z_w(m_i).$$

Therefore, we have

$$\sqrt[k]{Z_w\left(\prod_{i=1}^{k} m_i\right)} = \sqrt[k]{\prod_{i=1}^{k} Z_w(m_i)} < \frac{\sum_{i=1}^{k} Z_w(m_i)}{k} < \prod_{i=1}^{k} Z_w(m_i) = Z_w\left(\prod_{i=1}^{k} m_i\right).$$

(b). If $(m_i,\ m_j) > 1,\ i,\ j = 1,\ 2,\ \cdots,\ k$, and $i \neq j$, then let $m_i = p_1^{\alpha_{i1}} p_2^{\alpha_{i2}} \cdots p_r^{\alpha_{ir}}$, $\alpha_{is} \geq 0,\ i = 1,\ 2,\ \cdots,\ k;\ s = 1,\ 2,\ \cdots,\ r$. we have $Z_w(m_i) = p_1^{\beta_{i1}} p_2^{\beta_{i2}} \cdots p_r^{\beta_{ir}}$, where

$$\beta_{is} = \begin{cases} 0, & \text{if } \alpha_{is} = 0\ ; \\ 1, & \text{if } \alpha_{is} \geq 1. \end{cases}$$

Thus

$$\frac{\sum_{i=1}^{k} Z_w(m_i)}{k} = \frac{p_1^{\beta_{11}} p_2^{\beta_{12}} \cdots p_r^{\beta_{1r}} + p_1^{\beta_{21}} p_2^{\beta_{22}} \cdots p_r^{\beta_{2r}} + \cdots + p_1^{\beta_{k1}} p_2^{\beta_{k2}} \cdots p_r^{\beta_{kr}}}{k}$$

$$\leq \frac{p_1 p_2 \cdots p_r + p_1 p_2 \cdots p_r + \cdots + p_1 p_2 \cdots p_r}{k} = p_1 p_2 \cdots p_r = Z_w\left(\prod_{i=1}^{k} m_i\right),$$

and equality holds if and only if $\alpha_{is} \geq 1$, $i = 1,\ 2,\ \cdots,\ k$, $s = 1,\ 2,\ \cdots,\ r$.

$$\sqrt[k]{Z_w\left(\prod_{i=1}^{k} m_i\right)} = \sqrt[k]{p_1 p_2 \cdots p_r} \leq \sqrt[k]{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}}$$

$$\leq \frac{p_1^{\beta_{11}} p_2^{\beta_{12}} \cdots p_r^{\beta_{1r}} + p_1^{\beta_{21}} p_2^{\beta_{22}} \cdots p_r^{\beta_{2r}} + \cdots + p_1^{\beta_{k1}} p_2^{\beta_{k2}} \cdots p_r^{\beta_{kr}}}{k} = \frac{\sum_{i=1}^{k} Z_w(m_i)}{k},$$

where $\alpha_s = \sum_{i=1}^{k} \beta_{is}$, $s = 1,\ 2,\ \cdots,\ r$, but in this case, two equal sign in the above can't be hold in the same time.

So, we obtain

$$\sqrt[k]{Z_w\left(\prod_{i=1}^{k} m_i\right)} < \frac{\sum_{i=1}^{k} Z_w(m_i)}{k}.$$

From (a) and (b) we have

$$\sqrt[k]{Z_w\left(\prod_{i=1}^{k} m_i\right)} < \frac{\sum_{i=1}^{k} Z_w(m_i)}{k} \leq Z_w\left(\prod_{i=1}^{k} m_i\right),$$

and the equality holds if and only if all $m_1,\ m_2,\ \cdots,\ m_k$ have the same prime divisors. This proves Theorem 1.

To complete the proof of Theorem 2, we need the famous Vinogradov's three-primes theorem, which was stated as follows:

Lemma 1. Every odd integer bigger than c can be expressed as a sum of three odd primes, where c is a constant large enough.

Proof. (See reference [5]).

Lemma 2. Let $k \geq 3$ be an odd integer, then any sufficiently large odd integer n can be expressed as a sum of k odd primes

$$n = p_1 + p_2 + \cdots + p_k.$$

Proof. (See reference [6]).

Now we use these two Lemmas to prove Theorem 2. From Lemma 2 we know that for any odd integer $k \geq 3$, every sufficient large prime p can be expressed as

$$p = p_1 + p_2 + \cdots + p_k.$$

By the definition of $Z_w(n)$ we know that $Z_w(p) = p$. Thus,

$$\begin{aligned} & Z_w(p_1) + Z_w(p_2) + \cdots + Z_w(p_k) = p_1 + p_2 + \cdots + p_k = p = Z_w(p) \\ = \quad & Z_w(p_1 + p_2 + \cdots + p_k). \end{aligned}$$

This means that Theorem 2 is true for odd integer $k \geq 3$.

If $k \geq 4$ is an even number, then for every sufficient large prime p, $p-2$ is an odd number, and by Lemma 2 we have

$$p - 2 = p_1 + p_2 + \cdots + p_{k-1} \quad \text{or} \quad p = 2 + p_1 + p_2 + \cdots + p_{k-1}.$$

Therefore,

$$\begin{aligned} & Z_w(2) + Z_w(p_1) + Z_w(p_2) + \cdots + Z_w(p_{k-1}) = 2 + p_1 + p_2 + \cdots + p_{k-1} = p \\ = \quad & Z_w(p) = Z_w(2 + p_1 + p_2 + \cdots + p_{k-1}). \end{aligned}$$

This means that Theorem 2 is true for even integer $k \geq 4$.

At last, for any prime $p \geq 3$, we have

$$Z_w(p) + Z_w(p) = p + p = 2p = Z_w(2p),$$

so Theorem 2 is also true for $k = 2$. This completes the proof of Theorem 2.

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On the Smarandache function and the Fermat number

Jinrui Wang

Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China

Abstract For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : \ n \mid m!, \ n \in N\}$. The main purpose of this paper is using the elementary method to study the estimate problem of $S(F_n)$, and give a sharper lower bound estimate for it, where $F_n = 2^{2^n} + 1$ is called the Fermat number.

Keywords F. Smarandache function, the Fermat number, lower bound estimate, elementary method.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : \ n \mid m!, \ n \in N\}$. For example, the first few values of $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $S(11) = 11$, $S(12) = 4$, $\cdots$. About the elementary properties of $S(n)$, many authors had studied it, and obtained some interesting results, see references [1], [2], [3], [4] and [5]. For example, Lu Yaming [2] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any positive integer $k \geq 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite group positive integer solutions $(m_1, m_2, \cdots, m_k)$.

Dr. Xu Zhefeng [3] studied the value distribution problem of $S(n)$, and proved the following conclusion:

Let $P(n)$ denotes the largest prime factor of n, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} (S(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ denotes the Riemann zeta-function.

Chen Guohui [4] studied the solvability of the equation

$$S^2(x) - 5S(x) + p = x, \tag{1}$$

and proved the following conclusion:

Let p be a fixed prime. If $p = 2$, then the equation (1) has no positive integer solution; If $p = 3$, then the equation (1) has only one positive integer solution $x = 9$; If $p = 5$, then the equation (1) has only two positive integer solutions $x = 1,\ 5$; If $p = 7$, then the equation (1) has only two positive integer solutions $x = 21,\ 483$. If $p \geq 11$, then the equation (1) has only one positive integer solution $x = p(p-4)$.

Le Maohua [5] studied the lower bound of $S(2^{p-1}(2^p-1))$, and proved that for any odd prime p, we have the estimate:

$$S\left(2^{p-1}(2^p-1)\right) \geq 2p+1.$$

Recently, in a still unpublished paper, Su Juanli improved the above lower bound as $6p+1$. That is, she proved that for any prime $p \geq 7$, we have the estimate

$$S\left(2^{p-1}(2^p-1)\right) \geq 6p+1.$$

The main purpose of this paper is using the elementary method to study the estimate problem of $S(F_n)$, and give a sharper lower bound estimate for it, where $F_n = 2^{2^n}+1$ is the Fermat number. That is, we shall prove the following:

Theorem. For any positive integer $n \geq 3$, we have the estimate

$$S(F_n) \geq 8 \cdot 2^n + 1,$$

where $F_n = 2^{2^n}+1$ is called the Fermat number.

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem directly. First note that the Fermat number $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 65537$, they are all prime. So for $n = 3$ and 4, we have $S(F_3) = 257 \geq 8 \cdot 2^3 + 1$, $S(F_4) = 65537 > 8 \cdot 2^4 + 1$. Now without loss of generality we can assume that $n \geq 5$. If F_n be a prime, then from the properties of $S(n)$ we have $S(F_n) = F_n = 2^{2^n}+1 \geq 8 \cdot 2^n + 1$. If F_n be a composite number, then let p be any prime divisor of F_n, it is clear that $(2,\ p) = 1$. Let m denotes the exponent of 2 modulo p. That is, m denotes the smallest positive integer r such that

$$2^r \equiv 1 \pmod{p}.$$

Since $p \mid F_n$, so we have $F_n = 2^{2^n}+1 \equiv 0 \pmod{p}$ or $2^{2^n} \equiv -1 \pmod{p}$, and $2^{2^{n+1}} \equiv 1 \pmod{p}$. From this and the properties of exponent (see Theorem 10.1 of reference [6]) we have $m \mid 2^{n+1}$, so m is a divisor of 2^{n+1}. Let $m = 2^d$, where $1 \leq d \leq n+1$. It is clear that $p \nmid 2^d - 1$, if $d \leq n$. So $m = 2^{n+1}$ and $m \mid \phi(p) = p-1$. Therefore, $2^{n+1} \mid p-1$ or

$$p = h \cdot 2^{n+1} + 1. \tag{2}$$

Now we discuss the problem in following three cases:

(A) If F_n has more than or equal to three distinct prime divisors, then note that $2^{n+1}+1$ and $2\cdot 2^{n+1}+1$ can not be both primes, since one of them can be divided by 3. So from (2) we know that in all prime divisors of F_n, there exists at least one prime divisor p_i such that $p_i = h_i \cdot 2^{n+1}+1 \geq 4\cdot 2^{n+1}+1 = 8\cdot 2^n+1$.

(B) If F_n has just two distinct prime divisors, without loss of generality we can assume

$$F_n = \left(2^{n+1}+1\right)^{\alpha}\cdot\left(3\cdot 2^{n+1}+1\right)^{\beta} \quad \text{or} \quad F_n = \left(2\cdot 2^{n+1}+1\right)^{\alpha}\cdot\left(3\cdot 2^{n+1}+1\right)^{\beta}.$$

If $F_n = \left(2^{n+1}+1\right)^{\alpha}\cdot\left(3\cdot 2^{n+1}+1\right)^{\beta}$, and $\alpha\geq 4$ or $\beta\geq 2$, then from the properties of $S(n)$ we have the estimate

$$\begin{aligned} S(F_n) &\geq \max\left\{S\left(\left(2^{n+1}+1\right)^{\alpha}\right),\ S\left(\left(3\cdot 2^{n+1}+1\right)^{\beta}\right)\right\} \\ &= \max\left\{\alpha\cdot\left(2^{n+1}+1\right),\ \beta\cdot\left(3\cdot 2^{n+1}+1\right)\right\} \\ &\geq 8\cdot 2^n+1. \end{aligned}$$

If $F_n = 2^{2^n}+1 = \left(2^{n+1}+1\right)\cdot\left(3\cdot 2^{n+1}+1\right) = 3\cdot 2^{2n+2}+2^{n+3}+1$, then note that $n\geq 5$, we have the congruence

$$0\equiv 2^{2^n}+1-1 = 3\cdot 2^{2n+2}+2^{n+3}\equiv 2^{n+3} \pmod{2^{n+4}}.$$

This is impossible.

If $F_n = 2^{2^n}+1 = \left(2^{n+1}+1\right)^2\cdot\left(3\cdot 2^{n+1}+1\right) = 3\cdot 2^{3n+3}+3\cdot 2^{2n+3}+3\cdot 2^{n+1}+2^{2n+2}+2^{n+2}+1$, then we also have

$$0\equiv 2^{2^n}+1-1 = 3\cdot 2^{3n+3}+3\cdot 2^{2n+3}+3\cdot 2^{n+1}+2^{2n+2}+2^{n+2}\equiv 3\cdot 2^{n+1} \pmod{2^{n+2}}.$$

This is still impossible.

If $F_n = 2^{2^n}+1 = \left(2^{n+1}+1\right)^3\cdot\left(3\cdot 2^{n+1}+1\right)$, then we have

$$2^{2^n}+1\equiv\left(3\cdot 2^{n+1}+1\right)^2\equiv 3\cdot 2^{n+2}+1 \pmod{2^{n+4}}$$

or

$$0\equiv 2^{2^n}\equiv\left(3\cdot 2^{n+1}+1\right)^2-1\equiv 3\cdot 2^{n+2} \pmod{2^{n+4}}.$$

Contradiction with $2^{n+4}\ \dagger\ 3\cdot 2^{n+2}$.

If $F_n = \left(2\cdot 2^{n+1}+1\right)^{\alpha}\cdot\left(3\cdot 2^{n+1}+1\right)^{\beta}$, and $\alpha\geq 2$ or $\beta\geq 2$, then from the properties of $S(n)$ we have the estimate

$$\begin{aligned} S(F_n) &\geq \max\left\{S\left(\left(2\cdot 2^{n+1}+1\right)^{\alpha}\right),\ S\left(\left(3\cdot 2^{n+1}+1\right)^{\beta}\right)\right\} \\ &= \max\left\{\alpha\cdot\left(2\cdot 2^{n+1}+1\right),\ \beta\cdot\left(3\cdot 2^{n+1}+1\right)\right\} \\ &\geq 8\cdot 2^n+1. \end{aligned}$$

If $F_n = 2^{2^n}+1 = \left(2\cdot 2^{n+1}+1\right)\cdot\left(3\cdot 2^{n+1}+1\right)$, then we have

$$F_n = 2^{2^n}+1 = 3\cdot 2^{2n+3}+5\cdot 2^{n+1}+1.$$

From this we may immediately deduce the congruence

$$0\equiv 2^{2^n} = 3\cdot 2^{2n+3}+5\cdot 2^{n+1}\equiv 5\cdot 2^{n+1} \pmod{2^{2n+3}}.$$

This is not possible.

(C) If F_n has just one prime divisor, we can assume that

$$F_n = \left(2^{n+1}+1\right)^{\alpha} \quad \text{or} \quad F_n = \left(2\cdot 2^{n+1}+1\right)^{\alpha} \quad \text{or} \quad F_n = \left(3\cdot 2^{n+1}+1\right)^{\alpha}.$$

If $F_n = \left(2^{n+1}+1\right)^{\alpha}$, then it is clear that our theorem holds if $\alpha \geq 4$. If $\alpha = 1$, 2 or 3, then from the properties of the congruence we can deduce that $F_n = \left(2^{n+1}+1\right)^{\alpha}$ is not possible.

If $F_n = \left(2\cdot 2^{n+1}+1\right)^{\alpha}$ or $\left(3\cdot 2^{n+1}+1\right)^{\alpha}$, then our theorem holds if $\alpha \geq 2$. If $\alpha = 1$, then F_n be a prime, so our theorem also holds.

This completes the proof of Theorem.

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On the solvability of an equation involving the Smarandache function and Euler function

Weiguo Duan[†‡] and Yanrong Xue[†]

† Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China
‡ Department of Mathematics, Weinan Teacher's College, Weinan, Shaanxi, P.R.China

Abstract For any positive integer n, let $\phi(n)$ and $S(n)$ be the Euler function and the Smarandache function respectively. In this paper, we use the properties and the curve figure of these two functions to study the solvability of the equation $\sum_{i=1}^{n} S(i) = \phi(\frac{n(n+1)}{2})$, and prove that this equation has only two positive integer solutions $n = 1, 10$.

Keywords Euler function, F. Smarandache function, equation, solvability.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that n divides $m!$. That is, $S(n) = \min\{m : \ m \in N, \ n|m!\}$, where N denotes the set of all positive integers. From the definition of $S(n)$, it is easy to see that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of n into prime powers, then we have

$$S(n) = \max_{1 \le i \le k} \{S(p_i^{\alpha_i})\}.$$

It is clear that from this properties we can calculate the value of $S(n)$, the first few values of $S(n)$ are: $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $\cdots$. About the arithmetical properties of $S(n)$, some authors had studied it, and obtained many interesting results. For example, Lu Yaming [2] studied the solvability of an equation involving the F.Smarandache function $S(n)$, and proved that for any positive integer $k \ge 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite group positive integer solutions $(m_1, \ m_2, \ \cdots, \ m_k)$.

Jozsef Sandor [3] proved that for any positive integer $k \ge 2$, there exist infinite group positive integers $(m_1, \ m_2, \ \cdots, \ m_k)$ satisfying the inequality:

$$S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).$$

Also, there exist infinite group positive integers $(m_1, \ m_2, \ \cdots, \ m_k)$ such that

$$S(m_1 + m_2 + \cdots + m_k) < S(m_1) + S(m_2) + \cdots + S(m_k).$$

Rongji Chen [5] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any fixed $r \in N$ with $r \geq 3$, the positive integer n is a solution of

$$S(n)^r + S(n)^{r-1} + \cdots + S(n) = n$$

if and only if

$$n = p(p^{r-1} + p^{r-2} + \cdots + 1),$$

where p is an odd prime satisfying $p^{r-1} + p^{r-2} + \cdots + 1 | (p-1)!$.

Xiaoyan Li and Yanrong Xue [6] proved that for any positive integer k, the equation $S(n)^2 + S(n) = kn$ has infinite positive integer solutions, and each solution n has the form $n = pn_1$, where $p = kn_1 - 1$ is a prime.

For any positive integer n, the Euler function $\phi(n)$ is defined as the number of all positive integers not exceeding n, which are relatively prime to n. It is clear that $\phi(n)$ is a multiplicative function.

In this paper, we shall use the elementary method and compiler program to study the solvability of the equation:

$$S(1) + S(2) + \cdots + S(n) = \phi\left(\frac{n(n+1)}{2}\right), \tag{1}$$

and give its all positive integer solutions. That is, we shall prove the following:

Theorem. The equation

$$S(1) + S(2) + \cdots + S(n) = \phi\left(\frac{n(n+1)}{2}\right)$$

has and only has two positive integer solutions $n = 1,\ 10$.

§2. Main lemmas

In this section, we shall give two simple lemmas which are necessary in the proof of our Theorem. First we have the following:

Lemma 1. For any positive integer $n > 100$, we have the inequality

$$\sum_{i=1}^{n} S(i) \leqslant \frac{\pi^2}{11.99} \cdot \frac{n^2}{\ln n}.$$

Proof. From the mean value formula of $S(n)$ (See reference [7])

$$\sum_{n \leqslant x} S(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \mathrm{O}\left(\frac{x^2}{\ln^2 x}\right)$$

we know that there exists one constant $N > 0$ such that

$$\sum_{i=1}^{n} S(i) \leqslant \frac{\pi^2}{12} \cdot \frac{n^2}{\ln n} + \frac{1}{1199} \cdot \frac{\pi^2}{12} \cdot \frac{n^2}{\ln n} \leqslant \frac{\pi^2}{11.99} \cdot \frac{n^2}{\ln n}$$

holds for all positive integer $n > N$. We can take $N = 100$ by calculation. This completes the proof of Lemma 1.

Lemma 2. For Euler function $\phi(n)$, we have the estimate

$$\phi\left(\frac{n(n+1)}{2}\right) > \frac{n(n+1)}{4} \cdot e^{\frac{3}{4}} \cdot \frac{1}{\ln^{1.5}\left(2\ln\frac{n(n+1)}{2}\right)}.$$

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of n into prime powers, then there always exist some primes $p_1,\ p_2,\ \cdots p_s$ such that $p_1 p_2 \cdots p_s > n$. From [1] we have

$$\sum_{p \leqslant x} \ln p = x + \mathrm{O}\left(\frac{x}{\log x}\right),$$

by this estimate we know that

$$\ln n < \sum_{i=1}^{s} \ln p_i \leqslant \sum_{p_i \leqslant p_s} \ln p_i \leqslant p_s < 2\ln n.$$

Thus

$$\sum_{p|n} \frac{1}{p} \leqslant \sum_{p_i \leqslant p_s} \frac{1}{p_i} \leqslant \ln\ln p_s < \ln\ln(2\ln n).$$

Note that $\phi(n) = n\prod\limits_{p|n}\left(1-\frac{1}{p}\right)$, if $\frac{n(n+1)}{2}$ is even, then

$$\begin{aligned}
\phi\left(\frac{n(n+1)}{2}\right) &= \frac{n(n+1)}{2}\prod_{p|\frac{n(n+1)}{2}}\left(1-\frac{1}{p}\right) \\
&= \frac{n(n+1)}{4} e^{\sum\limits_{p|\frac{n(n+1)}{2},p\neq 2} \ln(1-\frac{1}{p})+\frac{1.5}{p}-\frac{1.5}{p}} \\
&= \frac{n(n+1)}{4} e^{-\sum\limits_{p|\frac{n(n+1)}{2},p\neq 2}\frac{1.5}{p}+\sum\limits_{p|\frac{n(n+1)}{2},p\neq 2}[\ln(1-\frac{1}{p})+\frac{1.5}{p}]} \\
&= \frac{n(n+1)}{4} e^{-\sum\limits_{p|\frac{n(n+1)}{2},p\neq 2}\frac{1.5}{p}} \\
&\geqslant \frac{n(n+1)}{4} \cdot e^{\frac{3}{4}} \cdot e^{-1.5\ln\ln(2\ln\frac{n(n+1)}{2})} \\
&> \frac{n(n+1)}{4} \cdot e^{\frac{3}{4}} \cdot \frac{1}{\ln^{1.5}\left(2\ln\frac{n(n+1)}{2}\right)}.
\end{aligned}$$

If $\frac{n(n+1)}{2}$ is odd, we can also get the same result. This completes the proof of Lemma 1.

§3. Proof of the theorem

In this section, we shall complete the proof of our Theorem. First we study the tendency of the functional digraph

$$f(x) = \frac{x(x+1)}{4} \cdot e^{\frac{3}{4}} \frac{1}{\ln^{1.5}(2\ln\frac{x(x+1)}{2})} - \frac{\pi^2}{11.99} \cdot \frac{x^2}{\ln x}.$$

By use of Mathematica compiler program we find that $f(x) > 0$, if $x > 100754$.

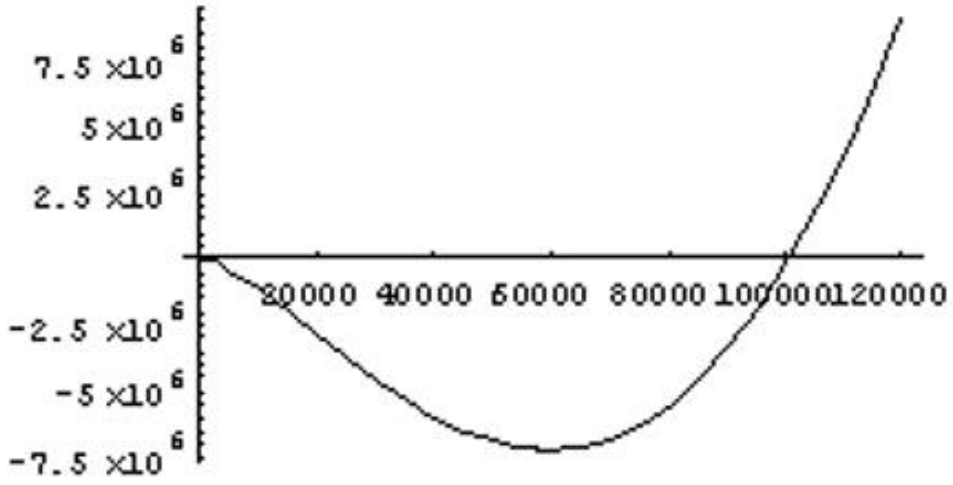

figure 1

From the figure 1 we know that if $n > 100754$, then

$$\sum_{i=1}^{n} S(i) \leqslant \frac{\pi^2}{11.99} \cdot \frac{n^2}{\ln n} < \frac{n(n+1)}{4} \cdot e^{\frac{3}{4}} \cdot \frac{1}{\ln^{1.5}(2\ln\frac{n(n+1)}{2})} < \phi\left(\frac{n(n+1)}{2}\right). \tag{2}$$

If $x \in (100754, +\infty)$, we use Mathematica compiler program to compute $f^{'}(x)$, then we find that the derivative $f^{'}(x)$ is positive, so (2) is also true if $x > 100754$.

Now we consider the solution of (1) for all $n \in [1,\ 100754]$. By use of the computer programming language, we obtain that the equation (1) has no any other positive integer solutions except $n = 1,\ n = 10$. This completes the proof of Theorem.

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The computing programme is given as follows if $n \in [1, 100754]$.

```
# include "stdio.h"
# include "math.h"
# define N 100754
int S(int n)
{int ret=1,num=n;
unsigned long int nn=1;
for(ret=1;ret<=n;ret++) { nn=nn*ret;
if(nn%num==0) break;} if (ret>n) ret=n;
return ret; }
int SumS(int n)
{int ret=0,i;
for(i=1;i<=n;i++) ret+=S(i);
return ret;}
int coprime(int i,int n)
{ int a=n,b=i;
while(a!=b) { if(a==0) return b;
if(b==0) return a;
if(a>b) a=a%b;
else
b=b%a;}
return a; }
int Euler(int n)
{int ret=1,i;
for(i=2;i<n;i++) {if(coprime(i,n)==1) ret++;} return ret;}
main()
{ int kk;
for(kk=1;kk<=N;kk++) if(SumS(kk)==Euler((kk*(kk+1)/2)))
printf( "rusult is % d\n" ,kk);
getch (); }
```

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The generalized Hermitian Procrustes problem and its approximation

Maolin Liang, Wansheng He and Lifang Dai

College of Mathematics and Statistics, Tianshui Normal University
Tianshui, Gansu, 741001, P. R. China
liangml2005@163.com

Abstract For orthogonal projective matrix R, i.e., $R^2 = R$ and $R^T = R$, we say that A is generalized Hermitian matrix, if $RAR = A^*$. In this paper, we investigate the least residual problem $\|AX - B\| = \min$ with given X, B, and associated optimal approximation problem in the generalized Hermitian matrix set. The general expressions of the solutions are derived by matrix decomposition.

Keywords Generalized Hermitian matrix, full-rank factorization, Procrustes problem, optimal approximation.

§1. Introduction

Some symbols and notations: Let $C_r^{m\times n}$ be the set of all $m\times n$ complex matrices with rank r, $HC^{n\times n}$ be the set of all $n\times n$ Hermitian matrices. Denoted by A^+, A^*, $rank(A)$ the Moore-Penrose generalized inverse, conjugate transpose, rank of matrix A, respectively. Moreover, I_n represents identity matrix of order n, and $J = (e_n, e_{n-1}, \cdots, e_1)$, $e_i \in C^n$ is the ith column of I_n. $\|\cdot\|$ stands for the Frobenius norm. Matrix $R \in C_r^{n\times n}$ is said to be projective (orthogonal projective) matrix, if $R^2 = R$ ($R^2 = R$ and $R^* = R$).

Definition 1.1. If $A \in C^{n\times n}$, we say that A is centro-symmetric matrix, if $JAJ = A$.

The centro-symmetric matrix has important and practical applications in information theory, linear system theory and numerical analysis (see [1-2]). As the extension of the centro-symmetric matrix, we define the following conception.

Definition 1.2. For given orthogonal projective matrix $R \in C_r^{n\times n}$, we say that $A \in C^{n\times n}$ is generalized Hermitian matrix, if $RAR = A^*$. Denote the set of all generalized Hermitian matrices by $GHC^{n\times n}$.

In this paper, we discuss two problems as follows:

Problem I.(Procrustes Problem): Given orthogonal projective matrix $R \in R^{n\times n}$, and $X, B \in C^{n\times m}$, find $A \in GHC^{n\times n}$ such that

$$\|AX - B\| = \min.$$

Problem II.(Optimal Approximation Problem): Given $M \in C^{n\times n}$, find $\hat{A} \in S_E$ such that

$$\| M - \hat{A}\| = \min_{A \in S_E} \|M - A\|,$$

where S_E is the solution set of Problem I.

Obviously, when $M = 0$, Problem II is changed into finding the least Frobenius norm solution of Problem I.

Many important results have been achieved about the above problems with different matrix sets, such as centro-symmetric matrix[3], symmetric matrix[4−5], R-symmetric matrix[6−7] and (R,S)-symmetric matrix[8] set. In this paper, we investigate the above problems in the generalized Hermitian matrix set by matrix decomposition.

§2. Preliminary knowledge

In this section, we discuss the properties and structures of (orthogonal) projective matrices $R \in C_r^{n\times n}$ and $A \in GHC^{n\times n}$.

Denote $s = rank(I-R)$, we know that $r+s = n$ since $R^2 = R$. Suppose that $p_1, p_2, \ldots, p_r$ and $q_1, q_2, \ldots, q_s$ are the normal orthogonal basis for range $\mathbf{R}(R)$ and null space $\mathbf{N}(R)$ of R, respectively. Let $P = (p_1, p_2, \ldots, p_r) \in C_r^{n\times r}$ and $Q = (q_1, q_2, \ldots, q_s) \in C_s^{n\times s}$, then

$$P^*P = I_r,\ Q^*Q = I_s, \tag{1}$$

$$RP = P,\ RQ = 0. \tag{2}$$

Lemma 2.1.(see [9]) Let matrix $A \in C_r^{n\times m}$ and its full-rank factorization $A = FG$, where $F \in C_r^{n\times r}$, $G \in C_r^{r\times m}$, then A is projective matrix if and only if $GF = I_r$.

Lemma 2.2. $R \in C_r^{n\times n}$ is projective matrix, then

$$R = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widehat{P} \\ \widehat{Q} \end{pmatrix}, \tag{3}$$

where matrix $\begin{pmatrix} P & Q \end{pmatrix}$ is invertible, and $\begin{pmatrix} P & Q \end{pmatrix}^{-1} = \begin{pmatrix} \widehat{P} \\ \widehat{Q} \end{pmatrix}$.

If R is orthogonal projective matrix, we have

$$R = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P^* \\ Q^* \end{pmatrix}, \tag{4}$$

where $\begin{pmatrix} P & Q \end{pmatrix}$ is unitary matrix.

Proof. Assume that the full-rank factorization of R is $R = P\widehat{P}$, we obtain from Lemma 2.1 and (1) that

$$\widehat{P} = P^*R,\ \widehat{P}P = I_r. \tag{5}$$

Similarly, if the full-rank factorization of $I - R$ is $I - R = Q\widehat{Q}$, we generate

$$\widehat{Q} = Q^*(I - R),\ \widehat{Q}Q = I_s, \tag{6}$$

since $(I - R)^2 = I - R$. Connecting with (1)(2)(5) and (6), we know that (3) holds. The equality (4) is obvious since $R^* = R$.

Lemma 2.3. Given matrices R as in (4) and $A \in GHC^{n\times n}$, then

$$A = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P^* \\ Q^* \end{pmatrix}, \forall\, G \in HC^{r\times r}. \tag{7}$$

Proof. According to Lemma 2.2 and Definition 2.1, it is clear that (7) holds.

Lemma 2.3 indicates that arbitrary matrix $M \in C^{n\times n}$ can be written as

$$M = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \begin{pmatrix} P^* \\ Q^* \end{pmatrix}.$$

§3. The solutions of Problem I and II

Given matrices $X, B \in C^{n\times m}$, partition

$$\begin{pmatrix} P^* \\ Q^* \end{pmatrix} X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \text{ and } \begin{pmatrix} P^* \\ Q^* \end{pmatrix} B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{8}$$

where $X_1, B_1 \in C^{r\times m}$ and $X_2, B_2 \in C^{s\times m}$.

We need the following two lemmas derived from References [7] and [8], respectively.

Lemma 3.1. Suppose that matrices X_1, B_1 in (8), then matrix equation $A_1X_1 = B_1$ is consistent for $A_1 \in HC^{r\times r}$, if and only if $B_1X_1^+X_1 = B_1$ and $X_1^*B_1 = B_1^*X_1$, the general solution is

$$A_1 = \tilde{A}_1 + (I_r - X_1X_1^+)K_1(I_r - X_1X_1^+),$$

where $\tilde{A}_1 = (I_r - \frac{X_1X_1^+}{2})B_1X_1^+ + (B_1X_1^+)^*(I_r - \frac{X_1X_1^+}{2}), \forall\, K_1 \in HC^{r\times r}$.

Lemma 3.2. Given matrices X_1, B_1 in (8), then

$$\min_{G\in C^{r\times r}} \parallel GX_1 - B_1 \parallel = \parallel B_1(I_r - X_1^+X_1) \parallel$$

if and only if $G = B_1X_1^+ + K_2(I_r - X_1X_1^+), \forall\, K_2 \in C^{r\times r}$.

According to Lemmas 3.1 and 3.2, we obtain

Lemma 3.3. For the above given matrices X_1, B_1,

$$\min_{A_1\in HC^{r\times r}} \parallel A_1X_1 - B_1 \parallel = \parallel B_1(I_r - X_1^+X_1) \parallel$$

if and only if

$$X_1^*B_1X_1^+ = X_1^+X_1B_1^*X_1X_1^+, \tag{9}$$

and the expression of A_1 is the same as that in Lemma 3.1.

Proof. $\parallel A_1X_1 - B_1 \parallel^2 = \parallel B_1 - B_1X_1^+X_1 + B_1X_1^+X_1 - A_1X_1 \parallel^2$

$$= \parallel B_1(I_r - X_1^+X_1) \parallel^2 + \parallel B_1X_1^+X_1 - A_1X_1 \parallel^2$$

Hence, the least residual can be attained only if $B_1X_1^+X_1 = A_1X_1$, which is consistent for $A_1 \in HC^{r\times r}$ under condition (9) by Lemma 3.3. The proof is completed.

Based on the previous analysis, Problem I can be solved in the following Theorem.

Theorem 3.1. Given matrix R as in (4), $X, B \in C^{n\times m}$ and the partition (8), then

$$\min_{A\in GHC^{n\times n}} \|AX - B\|^2 = \|B_1(I_r - X_1^+X_1)\|^2 + \|B_2\|^2, \tag{10}$$

if and only if (9) holds, at this time

$$A = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} \widehat{G} + (I_r - X_1X_1^+)K(I_r - X_1X_1^+) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P^* \\ Q^* \end{pmatrix}, \tag{11}$$

where $\tilde{G} = (I_r - \frac{X_1X_1^+}{2})B_1X_1^+ + (B_1X_1^+)^*(I_r - \frac{X_1X_1^+}{2})$, $\forall\, K \in HC^{r\times r}$.

Proof. According to the unitary invariance of Frobenius norm, formulas (4) and (7), we obtain

$$\begin{aligned} &\|AX - B\|^2 \\ &= \left\| \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P^* \\ Q^* \end{pmatrix} X - B \right\|^2 \\ &= \left\| \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} - \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right\|^2 \\ &= \|GX_1 - B_1\|^2 + \|B_2\|^2. \end{aligned}$$

Therefore, the problem (10) is equivalent to the following least residual problem

$$\min_{G\in HC^{r\times r}} \| GX_1 - B_1 \| .$$

From Lemma 3.3, we know that the minimum can be attained if and only if (9), and

$$G = \tilde{G} + (I_r - X_1X_1^+)K(I_r - X_1X_1^+),$$

where $K \in HC^{r\times r}$ is arbitrary. Submitting G into (7), then (11) holds.

The following lemma stated from [6].

Lemma 3.4. Let $L \in C^{q\times m}, \Delta \in C^{q\times q}$, $\Gamma \in C^{m\times m}$, and $\Delta^2 = \Delta = \Delta^*$, $\Gamma^2 = \Gamma = \Gamma^*$, then $\| L - \Delta L\Gamma \| = \min\limits_{N\in C^{q\times m}} \| L - \Delta N\Gamma \|$ if and only if $\Delta(L - N)\Gamma = 0$.

Let S_E be the solution set of Problem I. We can easily verify from its definition that S_E is a closed convex subsets in matrix space $C^{n\times n}$ under Frobenius norm. The optimal approximation theorem[10] reveals that Problem II has unique solution, which can be expressed in the next theorem.

Theorem 3.2. Suppose that the given matrix in Problem II is

$$M = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \begin{pmatrix} P^* \\ Q^* \end{pmatrix} \in C^{n\times n}$$

then

$$\min_{A\in S_E} \| M - A \| \tag{12}$$

if and only if

$$A = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} \widehat{G} + (I_r - X_1X_1^+)\frac{M_1+M_1^*}{2}(I_r - X_1X_1^+) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P^* \\ Q^* \end{pmatrix}, \tag{13}$$

where $\widehat{G}$ is the same as that in Theorem 3.1.

Proof. By using the unitary invariance of Frobenius norm and Theorem 3.1, we obtain

$$\begin{aligned} \| M - A \|^2 &= \left\| \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} - \begin{pmatrix} \widehat{G} + (I_r - X_1X_1^+)K(I_r - X_1X_1^+) & 0 \\ 0 & 0 \end{pmatrix} \right\|^2 \\ &= \| (M_1 - \widehat{G}) - (I_r - X_1X_1^+)K(I_r - X_1X_1^+) \|^2 \\ &\quad + \| M_2 \|^2 + \| M_3 \|^2 + \| M_4 \|^2, \end{aligned}$$

then the problem (12) equals to solve the minimum problem

$$\min_{K\in HC^{r\times r}} \| (M_1 - \widehat{G}) - (I_r - X_1X_1^+)K(I_r - X_1X_1^+) \| .$$

Moreover, since $\|M_1\|^2 = \|\frac{M_1+M_1^*}{2}\|^2 + \|\frac{M_1-M_1^*}{2}\|^2$, hence the above minimum problem can be transformed equivalently as

$$\min_{K\in HC^{r\times r}} \| (\frac{M_1 + M_1^*}{2} - \widehat{G_1}) - (I_r - X_1X_1^+)K(I_r - X_1X_1^+) \| .$$

We further deduce from Lemma 3.4 that

$$(I_r - X_1X_1^+)K(I_r - X_1X_1^+) = (I_r - X_1X_1^+)\frac{M_1+M_1^*}{2}(I_r - X_1X_1^+), \tag{14}$$

submitting (14) into (11), we obtain (13).

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On Frenet apparatus of partially null curves in semi-euclidean space

Suha Yilmaz and Melih Turgut

Dokuz Eylul University Buca Educational Faculty
Dep.of Mathematics, 35160, Buca-Izmir, Turkey
E-mail: suha.yilmaz@yahoo.com melih.turgut@gmail.com

Abstract In this paper, defining a vector product in Semi-Euclidean space E_2^4, we present a method to calculate Frenet apparatus of Partially Null curves. Thereafter, in the same space, using presented method, we prove that Frenet apparatus of a partially null evolute curve can be formed by involute's Frenet apparatus.
Keywords Semi-euclidean space, partially null curves, frenet apparatus.

§1. Introduction

Suffice it to say that the many important results in the theory of the curves in E^3 were initiated by G. Monge; and G. Darboux pionneered the moving frame idea. Thereafter, F. Frenet defined his moving frame and his special equations which play important role in mechanics and kinematics as well as in differential geometry. E. Cartan opened door of notion of null curves (for more details see [2]). And, thereafter null curves deeply studied by W. B. Bonnor [7] in Minkowski space-time. In the same space, Frenet equations for some special null; *Partially and Pseudo Null* curves are given in [4]. By means of Frenet equations, in [3] authors gave characterizations of such kind null curves lying on the pseudo-hyperbolic space in E_1^4. In [6], authors defined a vector product and by this way, they presented a method to calculate Frenet apparatus of space-like curves with non-null frame vectors and time-like curves in Minkowski space-time. Additionally, in [5] authors defined Frenet equations of pseudo null and a partially null curves in Semi-Euclidean space E_2^4.

In this work, first we defined vector product in E_2^4 and then, using Frenet equations defined in [5], we present a method to determine Frenet apparatus of partially null curves in E_2^4. Moreover, we prove that Frenet apparatus of a partially null evolute curve can be formed by involute's Frenet apparatus in terms of presented method.

§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_2^4 are briefly presented(A more complete elementary treatment can be found in [1]).

Semi-Euclidean space E_2^4 is an Euclidean space E^4 provided with the standard flat metric given by

$$g = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2, \tag{1}$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_2^4. Since g is an indefinite metric, recall that a vector $v \in E_2^4$ can have one of the three causal characters; it can be space-like if $g(v,v) > 0$ or $v = 0$, time-like if $g(v,v) < 0$ and null (light-like) if $g(v,v)$=0 and $v \neq 0$. Similary, an arbitrary curve $\alpha = \alpha(s)$ in E_2^4 can be locally be space-like, time-like or null (light-like), if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$. Therefore, v is a unit vector if $g(v,v) = \pm 1$. Next, vectors v, w in E_2^4 are said to be orthogonal if $g(v,w) = 0$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$. The Lorentzian hypersphere of center $m = (m_1, m_2, m_3, m_4)$ and radius $r \in R^+$ in the space E_2^4 defined by

$$S_2^3(m,r) = \left\{\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in E_2^4 : g(\alpha - m, \alpha - m) = r^2\right\}. \tag{2}$$

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_2^4. Then T, N, B_1, B_2 are, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. Space-like or time-like curve $\alpha(s)$ is said to be parametrized by arclength function s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. For a partially null unit speed curve in E_2^4, following Frenet equations are given in [5]

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & -\epsilon_2\tau & 0 & -\sigma \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{3}$$

where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equations
$g(T,T) = \epsilon_1 = \pm 1, g(N,N) = \epsilon_2 = \pm 1$, whereby $\epsilon_1\epsilon_2 = -1$
$g(B_1, B_1) = g(B_2, B_2) = 0, g(B_1, B_2) = 1$.

And here, κ, τ and σ are first, second and third curvature of the curve α, respectively. The set $\{\kappa, \tau, \sigma, T(s), N(s), B_1(s), B_2(s)\}$ is called Frenet apparatus of the curves. Let φ and δ be partially null unit speed curves in E_2^4. φ is an involute of δ if φ lies on the tangent line to δ at $\delta(s_0)$ and the tangents to δ and φ at $\delta(s_0)$ and φ are perpendicular for each s_0. φ is an evolute of δ if δ is an involute of φ. And this curve couple defined by $\varphi = \delta + \lambda T$.

In [5] authors gave a characterization about partially null curves with the following statement.

Theorem2.1. A partially null unit speed curve $\varphi(s)$ in E_2^4 with curvatures $\kappa \neq 0$, $\tau \neq 0$ for each $s \in I \subset R$ has $\sigma = 0$ for each s.

§3. Vector product in semi-euclidean space E_2^4

Definition 3.1. Let $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ be vectors

in E_2^4. The vector product in E_2^4 is defined with the determinant

$$a \wedge b \wedge c = - \begin{vmatrix} -e_1 & -e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}, \tag{4}$$

where e_1, e_2, e_3 and e_4 are mutually orthogonal vectors (coordinate direction vectors) satisfying equations

$e_1 \wedge e_2 \wedge e_3 = e_4$, $e_2 \wedge e_3 \wedge e_4 = e_1$, $e_3 \wedge e_4 \wedge e_1 = -e_2$, $e_4 \wedge e_1 \wedge e_2 = -e_3$.

Proposition 3.2. Let $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$, $c = (c_1, c_2, c_3, c_4)$ and $d = (d_1, d_2, d_3, d_4)$ be vectors in E_2^4. From the definition of vector product, there is a property in the space E_2^4 such as

$$g(a \wedge b \wedge c, a) = g(a \wedge b \wedge c, b) = g(a \wedge b \wedge c, c) = 0. \tag{5}$$

Proof of above proposition is elementary. Using definition, it can be easily obtained.

§4. A method to calculate Frenet apparatus of partially null curves in E_2^4

Let $\alpha = \alpha(s)$ be a partially null unit speed curve in E_2^4. By means of Frenet equations and Theorem 2.1, let us calculate following differentiations respect to s.

$$\frac{d\alpha}{ds} = \alpha'(s) = T. \tag{6}$$

$$\frac{d^2\alpha}{ds^2} = \alpha''(s) = \kappa N. \tag{7}$$

$$\frac{d^3\alpha}{ds^3} = \alpha'''(s) = \kappa^2 T + \kappa' N + \kappa\tau B_1. \tag{8}$$

Using (7), we easily have first curvature and principal normal, respectively,

$$\kappa = \|\alpha''(s)\|, \tag{9}$$

$$N = \frac{\alpha''(s)}{\kappa}. \tag{10}$$

Considering (4), we form following expression

$$T \wedge N \wedge \alpha''' = - \begin{vmatrix} -T & -N & B_1 & B_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \kappa^2 & \kappa' & \kappa\tau & 0 \end{vmatrix} = \kappa\tau B_2. \tag{11}$$

Taking the norm of both sides, we get second curvature and second binormal as

$$B_2 = \frac{T \wedge N \wedge \alpha'''}{\|T \wedge N \wedge \alpha'''\|}, \tag{12}$$

$$\tau = \frac{\|T \wedge N \wedge \alpha'''\|}{\|\alpha''\|}. \tag{13}$$

And, last using vector product we have first binormal as follow:

$$B_1 = N \wedge T \wedge B_2. \tag{14}$$

§5. Determination of Frenet apparatus of partially null involute-evolute curve couples in E_2^4

Theorem 5.1. Let φ and δ be partially null unit speed curves in E_2^4 and φ be evolute of δ. The Frenet apparatus of φ ($\{T_\varphi, N_\varphi, B_{1\varphi}, B_{2\varphi}, \kappa_\varphi, \tau_\varphi, \sigma_\varphi\}$) can be formed by Frenet apparatus of δ ($\{T, N, B_1, B_2, \kappa, \tau, \sigma\}$).

Proof. From Theorem 2.1 we know that $\sigma_\varphi = \sigma = 0$. Then, considering definition we write that

$$\varphi = \delta + \lambda T. \tag{15}$$

Definition of involute-evolute yields that $T_\varphi \perp T$. Using this and differentiating (15) respect to s, we have

$$1 + \frac{d\lambda}{ds} = 0. \tag{16}$$

Thus, we easily find $\lambda = c - s$, where c is constant. Rewriting (15) and differentiating it, we get

$$T_\varphi \frac{ds_\varphi}{ds} = (c - s)\kappa N. \tag{17}$$

Taking the norm of both sides of (17), we obtain

$$T_\varphi = N \tag{18}$$

and

$$\left\| \frac{d\varphi}{ds} \right\| = \frac{ds_\varphi}{ds} = (c - s)\kappa. \tag{19}$$

Considering presented method we calculate following derivatives. (Here $'$ denotes derivative according to s)

$$\varphi'' = (c - s)\kappa^2 T + [(c - s)\kappa' - \kappa]\, N + (c - s)\kappa\tau B_1. \tag{20}$$

$$\left\{ \begin{array}{c} \varphi''' = \left[-2\kappa^2 + 3(c-s)\kappa\kappa'\right] T + \left[-2\kappa' + (c-s)\kappa^3 + (c-s)\kappa''\right] N \\ + \left[-2\kappa\tau + 2(c-s)\kappa'\tau + (c-s)\kappa\tau'\right] B_1 \end{array} \right\}. \tag{21}$$

(20) gives us first curvature and the principal normal of φ as

$$\kappa_\varphi = \sqrt{-(c-s)^2\kappa^4 - \left[(c-s)\kappa' - \kappa\right]^2 + (c-s)^2\kappa^2\tau^2} \tag{22}$$

and

$$N_\varphi = \frac{(c-s)\kappa^2 T + \left[(c-s)\kappa' - \kappa\right] N + (c-s)\kappa\tau B_1}{\sqrt{-(c-s)^2\kappa^4 - \left[(c-s)\kappa' - \kappa\right]^2 + (c-s)^2\kappa^2\tau^2}}. \tag{23}$$

The vector product of $T_\varphi \wedge N_\varphi \wedge \varphi'''$ implies that

$$T_\varphi \wedge N_\varphi \wedge \varphi''' = -\frac{(c-s)^2\kappa^4}{\kappa_\varphi}\left(\frac{\tau}{\kappa}\right)' B_2. \tag{24}$$

Taking the norm of (24) and considering (22), we have second curvature and second binormal of φ

$$\tau_\varphi = \frac{(c-s)^2\kappa^4}{\kappa_\varphi}\left(\frac{\tau}{\kappa}\right)' \tag{25}$$

and

$$B_{2\varphi} = B_2. \tag{26}$$

Let us form

$$N_\varphi \wedge T_\varphi \wedge B_{2\varphi} = -\frac{1}{\kappa_\varphi}\begin{vmatrix} -T & -N & B_1 & B_2 \\ (c-s)\kappa^2 & (c-s)\kappa' - \kappa & (c-s)\kappa\tau & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}. \tag{27}$$

And therefore, we have

$$B_{1\varphi} = \frac{(c-s)\kappa}{\kappa_\varphi}\left[-\tau T - \kappa B_1\right] \tag{28}$$

.

This result completes the proof.

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An identity involving Nörlund numbers and Stirling numbers of the first kind

Hui Luo and Guodong Liu

Department of Mathematics, Huizhou University,
Huizhou, Guangdong, 516015, P.R.China

Abstract In this paper we investigate the relation for the Bernoulli numbers of higher order and the Stirling numbers of the first kind, and establish an computational formula for the Nörlund numbers.

Keywords Nörlund numbers, the Bernoulli numbers of higher order, the Stirling numbers of the first kind.

§1. Introduction and result

The Nörlund numbers N_n and the Bernoulli polynomials $B_n^{(k)}(x)$ of order k are defined, respectively, by（see [1], [4], [7]）

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty} N_n \frac{t^n}{n!}, \tag{1}$$

and

$$\left(\frac{t}{e^t-1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}. \tag{2}$$

The numbers $B_n^{(k)} = B_n^{(k)}(0)$ are the Bernoulli numbers of order k，$B_n^{(1)} = B_n$ are the ordinary Bernoulli numbers, and $b_n = \frac{1}{n!}B_n^{(n)}(1)$ are the Bernoulli numbers of the second kind. By (1) and (2), we can get (see [4], [7])：

$$N_n = B_n^{(n)}. \tag{3}$$

Stirling numbers of the first kind $s(n,k)$ can be defined by means of (see [1], [3], [5])

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k, \tag{4}$$

or by the generating function

$$(\log(1+x))^k = k!\sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}. \tag{5}$$

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It follows from (4) or (5) that

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), \tag{6}$$

with $s(n,0) = 0(n > 0), s(n,n) = 1, s(n,1) = (-1)^{n-1}(n-1)!(n > 0), s(n,k) = 0(k > n$ or $k < 0)$.

Stirling numbers of the second kind $S(n,k)$ can be defined by means of (see [1], [3], [5])

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)(x-2)\cdots(x-k+1), \tag{7}$$

or by the generating function

$$(e^x - 1)^k = k! \sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!}. \tag{8}$$

It follows from (7) or (8) that

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \tag{9}$$

with $S(n,0) = 0(n > 0), S(n,n) = 1, S(n,1) = 1(n > 0), S(n,k) = 0(k > n$ or $k < 0)$.

Associated Stirling numbers of the first kind $d(n,k)$ and associated Stirling numbers of the second kind $b(n,k)$ are defined, respectively, by (see [1], [3])

$$(\log(1+x) - x)^k = k! \sum_{n=2k}^{\infty} (-1)^{n-k} d(n,k)\frac{x^n}{n!}, \tag{10}$$

and

$$(e^x - 1 - x)^k = k! \sum_{n=2k}^{\infty} b(n,k)\frac{x^n}{n!}. \tag{11}$$

It follows from (10) and (11) that

$$d(n,k) = (n-1)d(n-2,k-1) + (n-1)d(n-1,k), \tag{12}$$

with $d(n,0) = 0(n > 0), d(0,0) = 1, d(n,1) = (n-1)!(n > 1), d(n,k) = 0(2k > n$ or $k < 0)$.
and

$$b(n,k) = (n-1)b(n-2,k-1) + kb(n-1,k), \tag{13}$$

with $b(n,0) = 0(n > 0), b(0,0) = 1, b(n,1) = 1(n > 1), b(n,k) = 0(2k > n$ or $k < 0)$.

In [7] the following recurrence formulas for Nörlund numbers N_n are found:

$$(-1)^n \frac{N_n}{n} = 1 - \sum_{k=0}^{n-1} \frac{(-1)^k}{n-k+1}\frac{N_k}{k!}, \quad (-1)^n \frac{N_n}{n!} = \sum_{k=0}^{n} \binom{n}{k}\frac{N_k}{k!}.$$

Howard [2], on the other hand, obtained relationships between Nörlund numbers N_n and Stirling numbers of the first kind $s(n,k)$, and the Bernoulli numbers of the second kind b_n:

$$N_n = \sum_{k=0}^{n} \frac{(-1)^k s(n,k)}{k+1}, \qquad N_n = \frac{1}{n+1} \sum_{k=0}^{[n/2]} \frac{s(n+1,2k+1)}{k+1}$$

and

$$N_n = n!\sum_{k=0}^{n}(-1)^{n-k}b_k, \qquad n!b_n = N_n + nN_{n-1}.$$

In [6], Liu obtained some computational formulas for Nörlund numbers N_n:

$$N_n = n\cdot n!\sum_{k=0}^{n-1}(-1)^{n-1-k}\frac{(k-1)!d(n+k,k)}{(n+k)!},$$

$$N_n = \sum_{k=0}^{n}\frac{n!k!}{(n+k)!}s(n+k,n)S(n,k),$$

$$N_n = \sum_{k=0}^{n}(-1)^k\frac{n}{n+k}\binom{2n}{n+k}S(n+k,k),$$

$$N_n = \sum_{k=0}^{n}(-1)^k\frac{n}{n+k}b(n+k,k).$$

The main purpose of this paper is that to prove an computational formula for Nörlund numbers. That is, we shall prove the following main conclusion.

Theorem. Let $n \geq 1$ be integers. Then we have

$$N_n = \frac{n}{2^n(1-2^n)}\sum_{l=0}^{n-1}\frac{2^l s(n,n-l)}{n-l}\sum_{r=0}^{n}\binom{n}{r}r^{n-l}. \tag{14}$$

§2. Proof of the Theorem

Proof. Writing

$$\begin{aligned}\left(\frac{t}{e^t-1}\right)^k &= 2^{-k}\left(\frac{2t}{e^{2t}-1}\right)^k(e^t+1)^k\\ &= 2^{-k}\sum_{r=0}^{k}\binom{k}{r}\left(\frac{2t}{e^{2t}-1}\right)^k e^{rt},\end{aligned}$$

we form the Abel convolution of

$$\left(\frac{2t}{e^{2t}-1}\right)^k = \sum_{l=0}^{\infty}2^l B_l^{(k)}\frac{t^l}{l!}$$

and

$$e^{rt} = \sum_{m=0}^{\infty}r^m\frac{t^m}{m!}.$$

Then

$$\begin{aligned}\left(\frac{t}{e^t-1}\right)^k &= 2^{-k}\sum_{r=0}^{k}\binom{k}{r}\sum_{n=0}^{\infty}\sum_{l+m=n}2^l B_l^{(k)}r^m\binom{n}{l}\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(2^{-k}\sum_{l=0}^{n}\binom{n}{l}2^l B_l^{(k)}\sum_{r=0}^{k}\binom{k}{r}r^{n-l}\right)\frac{t^n}{n!},\end{aligned}$$

whence

$$B_n^{(k)} = 2^{-k} \sum_{l=0}^{n-1} \binom{n}{l} 2^l B_l^{(k)} \sum_{r=0}^{k} \binom{k}{r} r^{n-l} + 2^{n-k} B_n^{(k)} 2^k,$$

on separating the term with $l = n$. Hence

$$B_n^{(k)} = \frac{1}{2^k(1-2^n)} \sum_{l=0}^{n-1} \binom{n}{l} 2^l B_l^{(k)} \sum_{r=0}^{k} \binom{k}{r} r^{n-l}. \tag{15}$$

Using the known formula ([7])

$$B_l^{(n)} = \frac{l!(n-1-l)!}{(n-1)!} s(n, n-l) \quad (l < n), \tag{16}$$

in (15) completes the proof of Theorem.

Remark 1. Taking $k = 1$ in (15), we have

$$B_n = \frac{1}{2(1-2^n)} \sum_{l=0}^{n-1} \binom{n}{l} 2^l B_l. \tag{17}$$

Remark 2. Taking $k = n+1, n+2, \cdots$ in (15), and note that (16) and

$$B_n^{(n+1)} = (-1)^n n!,$$

$$B_n^{(n+2)} = (-1)^n n! \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right)$$

(see [7]), we have

$$\sum_{l=0}^{n-1} 2^l s(n+1, n+1-l) \sum_{r=0}^{n+1} \binom{n+1}{r} r^{n-l} = (-1)^n n! 2^{n+1}(1-2^n), \tag{18}$$

$$\sum_{l=0}^{n-1} \frac{2^l(n+1-l)}{n+1} s(n+2, n+2-l) \sum_{r=0}^{n+2} \binom{n+2}{r} r^{n-l}$$

$$= (-1)^n n! 2^{n+2}(1-2^n) \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right). \tag{19}$$

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The study of $\sigma-$index on $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs

Shengzhang Ren and Wansheng He

College of mathematics, Tianshui Normal University, Tianshui, Gansu, 741000, China
E−mail: Renshengzhang1980@163.com

Abstract A $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs be a graph abtained from S_k whose every one degree vertex attached one cycle $C_i(i = 1, 2, \cdots, k)$. In this paper, we determine the lower and the higher bound for the Merrifield−simmons index in $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs in terms of the order k, and characterize the $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs with the smallest and the largest Merrifield−simmons index.

Keywords $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs, $\sigma-$index or Merrifield−Simmons index.

§1. Introduction

Let $G = (V, E)$ be a simple connected graph with the vertex set $V(G)$ and the edge set $E(G)$. For any $v \in V$,$N_G(v) = \{u \mid uv \in E(G)\}$ denotes the neighbors of v, and $d_G(v) =$ $\mid N_G(v) \mid$ is the degree of v in G; $N_G[v] = \{v\} \cup N_G(v)$. A leaf is a vertex of degree one and a stem is a vertex adjacent to at least one leaf. Let $E^{'} \subseteq E(G)$, we denote by $G - E^{'}$ the subgraph of G obtained by deleting the edges of $E^{'}$.$W \subseteq V(G)$, $G - W$ denotes the subgraph of G obtained by deleting the vertices of W and the edges incident with them. If a graph G has components $G_1, G_2, \cdots, G_k$, then G is denoted by $\bigcup_{i=1}^{k} G_i$. P_n denotes the path on n vertices, C_n is the cycle on n vertices, and S_n is the star consisting of one center vertex adjacent to $n-1$ leaves and T_n is a tree on n vertices.

For a graph $G = (V, E)$, a subset $S \subseteq V$ is called independent if no two vertices of S are adjacent in G. The set of independent sets in G is denoted by $I(G)$. The empty set is an independent set. The number of independent sets in G, denoted by $\sigma-$index, is called the $Merrifield - Simmons$ index in theoratical chemistry.the $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs is abtained from S_k whose every one degree vertex attached one cycle $C_{s_i}(i = 1, \cdots, k)$.

The $Merrifield-Simmons$ index [1-3] is one of the topogical indices whose mathematical properties were studied in some detail [4-12] whereas its applicability for QSPR and QSAR was examined to a much lesser extent; in [2] it was shown that $\sigma-$index is correlated with the boiling points.

In this paper, we investigate the $Merrifield-Simmons$ index on $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs. We characterize the $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs with the smallest and the largest

Merrifield-Simmons index.

§2. Some known results

We give with several important lemmas from [2-6] will be helpful to the proofs of our main results, and also give three lemmas which will increase the Merrifield-Simmons index.

Lemma2.1.[2] Let G be a graph with k components $G_1, G_2, \cdots, G_k$, then

$$\sigma(G) = \prod_{i=1}^{k} \sigma(G_i).$$

Lemma 2.2.[4] For any graph G with any $v \in V(G)$, we have $\sigma(G) = \sigma(G-v)+\sigma(G-[v])$, where $[v] = N_G(v) \bigcup v$.

Lemma 2.3.[3] Let T be a tree, then $F_{n+2} \leq \sigma(T) \leq 2^{n-1}+1$ and $\sigma(T) = F_{n+2}$ if and only if $T \cong P_n$ and $\sigma(T) = 2^{n-1}+1$ if and only if $T \cong S_n$.

Lemma 2.4.[5] Let $n = 4m+i(i \in \{1,2,3,4\})$ and $m \geq 2$, then $\sigma((P_n, v_2, T)) > \sigma((P_n, v_4, T)) > \cdots > \sigma((P_n, v_{2m+2\rho}, T)) > \cdots > \sigma((P_n, v_{2m+1}, T)) > \sigma((P_n, v_3, T)) > \sigma((P_n, v_1, T))$, where $\rho = 0$ if $i = 1, 2$ and $\rho = 1$ if $i = 3, 4$.

Lemma 2.5.[6] Let $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ and by definition of Fibonacci number F_n and Lucas number L_n, we know

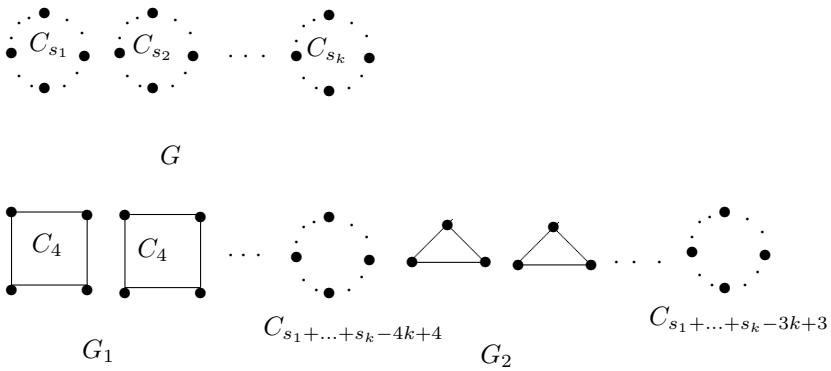
$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, L_n = \alpha^n + \beta^n, F_n \cdot F_m = \frac{1}{5}(L_{n+m} - (-1)^n \cdot L_{m-n}).$$

Lemma 2.6. Let G is $Q(S_3, C_s, C_l, C_h)$ graphs with n vertices, then $\sigma(Q(S_3, C_s, C_l, C_h)) = (F_{s+1}+F_{s-1})(F_{l+1}+F_{l-1})(F_{h+1}+F_{h-1}) + F_{s+1}F_{l+1}F_{h+1}$.

Proof. From the lemmas 2.1 and 2.2, we have

$$\sigma(Q(S_3, C_s, C_l, C_h)) = L_s L_l L_h + F_{s+1}F_{l+1}F_{h+1}.$$

From the lemma 2.6, we have $L_s = F_{s+1}+F_{s-1}, L_l = F_{l+1}+F_{l-1}, L_h = F_{h+1}+F_{h-1}$, so $\sigma(Q(S_3, C_s, C_l, C_h)) = (F_{s+1}+F_{s-1})(F_{l+1}+F_{l-1})(F_{h+1}+F_{h-1}) + F_{s+1}F_{l+1}F_{h+1}$.



Picture 2.1

Lemma 2.7. Let $G = C_{s_1} \cup C_{s_2} \cup \cdots \cup C_{s_k}$ graphs and $G_1 = C_4 \cup C_4 \cup \cdots \cup C_4 \cup C_{s_1+s_2+\cdots+s_k-4(k-1)}$ graph as shown picture 2.1 where $k-1$ are numbers of C_4, then $\sigma(G_1) \geq \sigma(G)$ and $\sigma(G_1) = \sigma(G)$ if and only if $G_1 \cong G$.

Proof. If $k = 2$, then

$$\begin{aligned}
\sigma(G) &= L_{s_1} L_{n-s_1} \\
&= (F_{s_1+1} + F_{s_1-1})(F_{n-s_1+1} + F_{n-s_1-1}) \\
&= F_{s_1+1}F_{n-s_1+1} + F_{s_1+1}F_{n-s_1-1} + F_{s_1-1}F_{n-s_1+1} + F_{s_1-1}F_{n-s_1-1} \\
&= \frac{1}{5}[(L_{n+2} + (-1)^{s_1}L_{n-2s_1}) + (L_n + (-1)^{s_1}L_{n-2s_1-2}) + (L_n + (-1)^{s_1}L_{n-2s_1+2}) \\
&\quad +(L_{n-2} + (-1)^{s_1}L_{n-2s_1})] \\
&= \frac{1}{5}[(L_{n+2} + 2L_n + L_{n-2}) + (-1)^{s_1}(2L_{n-2s_1} + L_{n-2s_1-2} + L_{n-2s_1+2})].
\end{aligned}$$

From above, we know that the result is correct if $K = 2$. We presume that the result is correct if $K = k$, then if $K = k+1$, we have

$$\begin{aligned}
\sigma(G) &= \prod_{i=1}^{k+1} \sigma(C_{s_i}) \\
&= \prod_{i=1}^{k} \sigma(C_{s_i}) \cdot \sigma(C_{s_{k+1}}) \\
&\leq L_4^{k-1} L_{s_1+s_2+\cdots+s_k-4(k-1)} L_{s_{k+1}} \\
\sigma(G_1) &= L_4^k L_{s_1+_{s_2}+\cdots+s_{k+1}-4k}. \\
\sigma(G_1) - \sigma(G) &\geq L_4^k L_{s_1+s_2+\cdots+s_{k+1}-4k} - L_4^{k-1} L_{s_1+s_2+\cdots+s_k-4(k-1)} L_{s_{k+1}} \\
&= L_4^{k-1}(L_4 L_{s_1+s_2+\cdots+s_{k+1}-4k} - L_{s_1+s_2+\cdots+s_k-4(k-1)} L_{s_{k+1}}) \\
&= L_4^{k-1}[(\alpha^4 + \beta^4)(\alpha^{s_1+s_2+\cdots+s_{k+1}-4k} + \beta^{s_1+s_2+\cdots+s_{k+1}-4k}) \\
&\quad -(\alpha^{s_1+s_2+\cdots+s_k-4(k-1)} + \beta^{s_1+s_2+\cdots+s_k-4(k-1)})(\alpha^{s_{k+1}} + \beta^{s_{k+1}})] \\
&= L_4^{k-1}(L_{s_1+s_2+\cdots+s_{k+1}-4k-4} - (-1)^{s_{k+1}} L_{s_1+s_2+\cdots+s_k-4k+4-s_{k+1}}) \\
&\geq 0.
\end{aligned}$$

From above, we know that the result is correct.

Lemma 2.8. Let $G = C_{s_1} \cup C_{s_2} \cup \cdots \cup C_{s_k}$ graphs and $G_2 = C_3 \cup C_3 \cup \cdots \cup C_3 \cup C_{s_1+s_2+\cdots+s_k-3(k-1)}$ graph as shown picture 2.1 where $k-1$ are numbers of C_3, then $\sigma(G) \geq \sigma(G_2)$ and $\sigma(G) = \sigma(G_2)$ if and only if $G_2 \cong G$.

Proof. If $k = 2$, from the lemma 2.7 we know that the result is correct.

We presume that the result is correct if $K = k$, then if $K = k+1$, we have

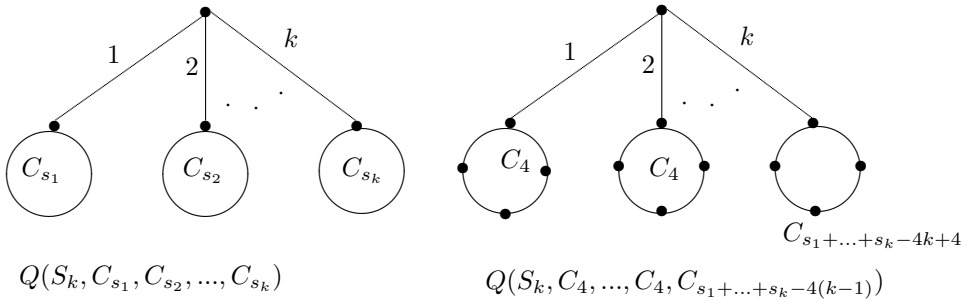
$$\begin{aligned}
\sigma(G) &= \prod_{i=1}^{k+1} \sigma(C_{s_i}) = \prod_{i=1}^{k} \sigma(C_{s_i}) \cdot \sigma(C_{s_{k+1}}) \geq L_3^{k-1} L_{s_1+s_2+\cdots+s_k-3(k-1)} L_{s_{k+1}} \\
\sigma(G_2) &= L_3^k L_{s_1+_{s_2}+\cdots+s_{k+1}-3k}. \\
\sigma(G) - \sigma(G_2) &\geq L_3^{k-1} L_{s_1+s_2+\cdots+s_k-3k+3} L_{s_{k+1}} - L_3^k L_{s_1+s_2+\cdots+s_{k+1}-3k} \\
&= L_3^{k-1}(L_{s_1+s_2+\cdots+s_k-3(k-1)} L_{s_{k+1}} - L_3 L_{s_1+s_2+\cdots+s_{k+1}-3k}) \\
&= L_3^{k-1}[(\alpha^{s_1+s_2+\cdots+s_k-3(k-1)} + \beta^{s_1+s_2+\cdots+s_k-3(k-1)})(\alpha^{s_{k+1}} + \beta^{s_{k+1}}) \\
&\quad -(\alpha^3 + \beta^3)(\alpha^{s_1+s_2+\cdots+s_{k+1}-3k} + \beta^{s_1+s_2+\cdots+s_{k+1}-3k})] \\
&= L_3^{k-1}(L_{s_1+s_2+\cdots+s_{k+1}-3k-3} + (-1)^{s_{k+1}} L_{s_1} + s_2 + \cdots + s_k - 3k - 3 - s_{k+1}) \\
&\geq 0.
\end{aligned}$$

From above, we know that the result is correct.

§3. The graph with the largest Merrifield-Simmons index in $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs

In this section, we will find the $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs with the largest $\sigma-$index in $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs. and give some good results on orders of $\sigma-$index.

Definition 3.1. Let $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs be a graph abtained from S_k whose every one degree vertex attached one cycle $C_i(i = 1, 2, \cdots, k)$ as shown Picture 3.1.



Picture 3.1

Remark: Above graphs will be used frequently in this paper.

Theorem 3.1. Let s is constant and $l = 4j + i, i \in \{1, 2, 3, 4\}$ and $j \geq 2$. Then
$\sigma(Q(S_3, C_s, C_4, C_{n-s-5})) > \sigma(Q(S_3, C_s, C_6, C_{n-s-7}))$
$> \cdots > \sigma(Q(S_3, C_s, C_{2j+2\rho}, C_{n-s-2j-2\rho-1})) > \sigma(Q(S_3, C_s, C_{2j+1}, C_{n-s-2j-2}))$
$> \cdots > \sigma(Q(S_3, C_s, C_5, C_{n-s-6})) > \sigma(Q(S_3, C_s, C_3, C_{n-s-4}))$,
where $\rho = 0$ if $i = 1, 2$ and $\rho = 1$ if $i = 3, 4$.

Proof. From Lemma 2.6, we have

$$\begin{aligned}
&\sigma(Q(S_3, C_s, C_l, C_{n-s-l-1})) \\
&= (F_{s+1} + F_{s-1})(F_{l+1} + F_{l-1})(F_{n-s-l} + F_{n-s-l-2}) + F_{s+1}F_{l+1}F_{n-s-l} \\
&= F_{s+1}F_{l+1}F_{n-s-l} + F_{s+1}F_{l+1}F_{n-s-l-2} + F_{s+1}F_{l-1}F_{n-s-l} + F_{s+1}F_{l-1}F_{n-s-l-2} \\
&\quad + F_{s-1}F_{l+1}F_{n-s-l} + F_{s-1}F_{l+1}F_{n-s-l-2} + F_{s-1}F_{l-1}F_{n-s-l} + F_{s-1}F_{l-1}F_{n-s-l-2}.
\end{aligned}$$

From Lemma 2.5, we have

$$\begin{aligned}
&\sigma(Q(S_3, C_s, C_l, C_{n-s-l-1})) \\
&= \frac{1}{5}[2F_{s+1}(L_{n-s+1} - (-1)^{l+1}L_{n-s-2l-1}) + F_{s+1}(L_{n-s-1} - (-1)^{l+1}L_{n-s-2l-3}) \\
&\quad + F_{s+1}(L_{n-s-1} - (-1)^{l-1}L_{n-s-2l+1}) + F_{s-1}(L_{n-s+1} - (-1)^{l+1}L_{n-s-2l-1}) \\
&\quad + F_{s-1}(L_{n-s-1} - (-1)^{l-1}L_{n-s-2l+1}) + F_{s-1}(L_{n-s-1} - (-1)^{l+1}L_{n-s-2l-3}) \\
&\quad + F_{s+1}(L_{n-s-3} - (-1)^{l-1}L_{n-s-2l-1}) + F_{s-1}(L_{n-s-3} - (-1)^{l-1}L_{n-s-2l-1})] \\
&= \frac{1}{5}[(2F_{s+1}L_{n-s+1}) + (2F_{s+1}L_{n-s-1}) + (F_{s-1}L_{n-s+1}) + (2F_{s-1}L_{n-s-1}) \\
&\quad + (F_{s+1}L_{n-s-3}) + (F_{s-1}L_{n-s-3}) + (-1)^l(3F_{s+1}L_{n-s-2l-1} + F_{s+1}L_{n-s-2l-3} \\
&\quad + 2F_{s-1}L_{n-s-2l-1} + F_{s-1}L_{n-s-2l+1} + F_{s-1}L_{n-s-2l-3} + F_{s+1}L_{n-s-2l+1})].
\end{aligned}$$

From above, we know that the result is correct.

Theorem 3.2. Let $s = 2j + i, i \in \{1, 2, 3, 4\}$ and $j \geq 2$, then
$\sigma(Q(S_3, C_4, C_4, C_{n-9})) > \sigma(Q(S_3, C_6, C_4, C_{n-11}))$
$> \cdots > \sigma(Q(S_3, C_{2j+2\rho}, C_4, C_{n-2j-2\rho-5})) > \sigma(Q(S_3, C_{2j+1}, C_4, C_{n-2j-6}))$

$> \cdots > \sigma(Q(S_3, C_5, C_4, C_{n-10})) > \sigma(Q(S_3, C_3, C_4, C_{n-8}))$,
where $\rho = 0$ if $i = 1, 2$ and $\rho = 1$ if $i = 3, 4$.

Proof. From lemma 2.6, we have

$$\begin{aligned}
&\sigma(Q(S_3, C_s, C_4, C_{n-s-5}))\\
&\quad = 2F_{s+1}F_5F_{n-s-4} + F_{s+1}F_5F_{n-s-6} + F_{s+1}F_3F_{n-s-4} + F_{s-1}F_5F_{n-s-4}\\
&\qquad +F_{s-1}F_3F_{n-s-4} + F_{s-1}F_5F_{n-s-6} + F_{s+1}F_3F_{n-s-6} + F_{s-1}F_3F_{n-s-6}\\
&\quad = 12F_{s+1}F_{n-s-4} + 7F_{s-1}F_{n-s-4} + 7F_{s+1}F_{n-s-6} + 7F_{s-1}F_{n-s-6}\\
&\quad = \frac{1}{5}[12(L_{n-3} - (-1)^{s+1}L_{n-2s-5}) + 7(L_{n-5} - (-1)^{s-1}L_{n-2s-3})\\
&\qquad +7(L_{n-5} - (-1)^{s+1}L_{n-2s-7}) + 7(L_{n-7} - (-1)^{s-1}L_{n-2s-5})]\\
&\quad = \frac{1}{5}[(12L_{n-3} + 14L_{n-5} + 7L_{n-7})\\
&\qquad +(-1)^s(19L_{n-2s-5} + 7L_{n-2s-3} + 7L_{n-2s-7})].
\end{aligned}$$

From above, we know that the result is correct.

Corollary 1. The $Q(S_3, C_s, C_l, C_{n-s-l-1})$ graphs with the largest $\sigma-$ index is $Q(S_3, C_4, C_4, C_{n-9})$.

Theorem 3.3. Let $n \geq 4k$, then
$\sigma(Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})) \leq \sigma(Q(S_k, C_4, \cdots, C_4, C_{s_1+s_2+\cdots+s_k-4k+4}))$ and
$\sigma(Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})) = \sigma(Q(S_k, C_4, \cdots, C_4, C_{s_1+s_2+\cdots+s_k-4k+4}))$ if only and if
$Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k}) \cong Q(S_k, C_4, \cdots, C_4, C_{s_1+s_2+\cdots+s_k-4k+4})$.

Proof. If $K = 3$, we have proofed that the result is correct. We presume that the result is correct, if $K = k$, then if $K = k + 1$ we have

$$\begin{aligned}
&\sigma(Q(S_{k+1}, C_{s_1}, C_{s_2}, \cdots, C_{s_{k+1}}))\\
&\quad = \sigma(Q(S_{k+1}, C_{s_1}, C_{s_2}, \cdots, C_{s_{k+1}}) - v) + \sigma(Q(S_{k+1}, C_{s_1}, C_{s_2}, \cdots, C_{s_{k+1}}) - [v])\\
&\quad \leq (L_4^{k-1}L_{s_1+s_2+\cdots+s_k-4(k-1)} + F_5^{k-1}F_{s_1+s_2+\cdots+s_k-4k+5})F_{s_{k+1}+1}\\
&\qquad +L_4^{k-1}L_{s_1+s_2+\cdots+s_k-4k+4}F_{s_{k+1}-1}\\
&\sigma(Q(S_{k+1}, C_4, \cdots, C_4, C_{s_1+s_2+\cdots+s_{k+1}-4k}))\\
&\quad = L_4^kL_{s_1+s_2+\cdots+s_{k+1}-4k} + F_5^kF_{s_1+s_2+\cdots+s_{k+1}-4k+1}\\
&\sigma(Q(S_{k+1}, C_4, \cdots, C_4, C_{s_1+s_2+\cdots+s_{k+1}-4k})) - \sigma(Q(S_{k+1}, C_{s_1}, C_{s_2}, \cdots, C_{s_{k+1}}))\\
&\quad \geq L_4^{k-1}[L_4L_{s_1+s_2+\cdots+s_{k+1}-4k} - L_{s_1+s_2+\cdots+s_k-4k+4}(F_{s_{k+1}+1} + F_{s_{k+1}-1})]\\
&\qquad +F_5^{k-1}(F_5F_{s_1+s_2+\cdots+s_{k+1}-4k+1} - F_{s_1+s_2+\cdots+s_k-4k+5}F_{s_{k+1}+1})\\
&\quad = L_4^{k-1}(L_{s_1+s_2+\cdots+s_{k+1}-4k-4} - (-1)^{s_{k+1}}L_{s_1+s_2+\cdots+s_k-4k+4-s_{k+1}})\\
&\qquad +F_5^{k-1}(L_{s_1+s_2+\cdots+s_{k+1}-4k-4} - (-1)^{s_{k+1}}L_{s_1+s_2+\cdots+s_k-4k+4-s_{k+1}})\\
&\quad = (L_4^{k-1} - F_5^{k-1})(L_{s_1+s_2+\cdots+s_{k+1}-4k-4} - (-1)^{s_{k+1}}L_{s_1+s_2+\cdots+s_k-4k+4-s_{k+1}}) \geq 0.
\end{aligned}$$

From above, we know that the result is correct.

§4. The graph with the smallest Merrifield-Simmons index in $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs

In this section, we will find the $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs with the smallest Merrifield-Simmons index.

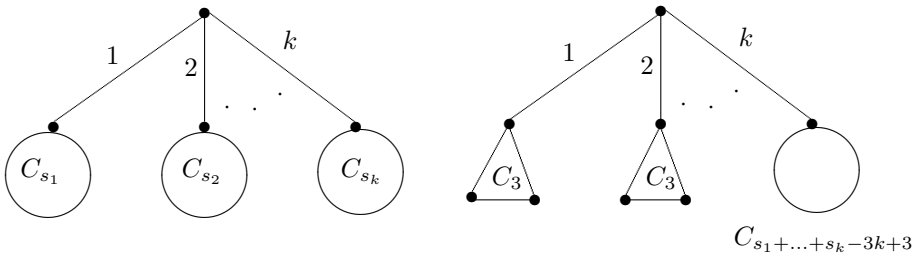
Theorem 4.1. Let $s = 2j + i, i \in \{1, 2, 3, 4\}$ and $j \geq 2$, then
$\sigma(Q(S_3, C_4, C_3, C_{n-8})) > \sigma(Q(S_3, C_6, C_3, C_{n-10}))$

$> \cdots > \sigma(Q(S_3, C_{2j+2\rho}, C_3, C_{n-2j-2\rho-4})) > \sigma(Q(S_3, C_{2j+1}, C_3, C_{n-2j-5}))$
$> \cdots > \sigma(Q(S_3, C_5, C_3, C_{n-9})) > \sigma(Q(S_3, C_3, C_3, C_{n-7}))$,
where $\rho = 0$ if $i = 1, 2$ and $\rho = 1$ if $i = 3, 4$.

Proof. From lemma 2.6, we have

$$\begin{aligned}
&\sigma(Q(S_3, C_s, C_3, C_{n-s-4})) \\
&\quad = 2F_{s+1}F_4F_{n-s-3} + F_{s+1}F_4F_{n-s-5} + F_{s+1}F_2F_{n-s-3} + F_{s-1}F_4F_{n-s-3} \\
&\qquad + F_{s-1}F_2F_{n-s-3} + F_{s-1}F_4F_{n-s-5} + F_{s+1}F_2F_{n-s-5} + F_{s-1}F_2F_{n-s-5} \\
&\quad = 7F_{s+1}F_{n-s-3} + 4F_{s+1}F_{n-s-5} + 4F_{s-1}F_{n-s-3} + 4F_{s-1}F_{n-s-5} \\
&\quad = \frac{1}{5}[7(L_{n-2} - (-1)^{s+1}L_{n-2s-4}) + 4(L_{n-4} - (-1)^{s-1}L_{n-2s-6}) \\
&\qquad + 4(L_{n-4} - (-1)^{s+1}L_{n-2s-2}) + 4(L_{n-6} - (-1)^{s-1}L_{n-2s-4})] \\
&\quad = \frac{1}{5}[(7L_{n-2} + 8L_{n-4} + 4L_{n-6}) \\
&\qquad + (-1)^s(11L_{n-2s-4} + 4L_{n-2s-6} + 4L_{n-2s-2})].
\end{aligned}$$

From above, we know that the result is correct.



$Q(S_k, C_{s_1}, C_{s_2}, ..., C_{s_k})$ $\quad$ $Q(S_k, C_3, ..., C_3, C_{s_1+\ldots+s_k-3(k-1)})$

Picture 4.1

Theorem 4.2. Let $n \geq 4k$, then
$\sigma(Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})) \geq \sigma(Q(S_k, C_3, \cdots, C_3, C_{s_1+s_2+\cdots+s_k-3k+3}))$ and
$\sigma(Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})) = \sigma(Q(S_k, C_3, \cdots, C_3, C_{s_1+s_2+\cdots+s_k-3k+3}))$
if only and if $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k}) \cong Q(S_k, C_3, C_3, \cdots, C_{s_1+s_2+\cdots+s_k-3k+3})$.

Proof. If $K = 3$, we have proofed that the result is correct. We presume that the result is correct, if $K = k$, then if $K = k + 1$ we have

$$\begin{aligned}
&\sigma(Q(S_{k+1}, C_{s_1}, C_{s_2}, \cdots, C_{s_{k+1}})) \\
&\quad = \sigma(Q(S_{k+1}, C_{s_1}, C_{s_2}, \cdots, C_{s_{k+1}}) - v) + \sigma(Q(S_{k+1}, C_{s_1}, C_{s_2}, \cdots, C_{s_{k+1}}) - [v]) \\
&\quad \geq (L_3^{k-1}L_{s_1+s_2+\cdots+s_k-3(k-1)} + F_4^{k-1}F_{s_1+s_2+\cdots+s_k-3k+4})F_{s_{k+1}+1} \\
&\qquad + L_3^{k-1}L_{s_1+s_2+\cdots+s_k-3k+3}F_{s_{k+1}-1} \\
&\sigma(Q(S_{k+1}, C_3, \cdots, C_3, C_{s_1+s_2+\cdots+s_{k+1}-3k})) \\
&\quad = L_3^kL_{s_1+s_2+\cdots+s_{k+1}-3k} + F_4^kF_{s_1+s_2+\cdots+s_{k+1}-3k+1} \\
&\sigma(Q(S_{k+1}, C_3, \cdots, C_3, C_{s_1+s_2+\cdots+s_{k+1}-3k})) - \sigma(Q(S_{k+1}, C_{s_1}, C_{s_2}, \cdots, C_{s_{k+1}})) \\
&\quad \leq L_3^{k-1}[L_3L_{s_1+s_2+\cdots+s_{k+1}-3k} - L_{s_1+s_2+\cdots+s_k-3k+3}(F_{s_{k+1}+1} + F_{s_{k+1}-1})] \\
&\qquad + F_4^{k-1}(F_4F_{s_1+s_2+\cdots+s_{k+1}-3k+1} - F_{s_1+s_2+\cdots+s_k-3k+4}F_{s_{k+1}+1}) \\
&\quad = L_3^{k-1}(-L_{s_1+s_2+\cdots+s_{k+1}-3k-3} - (-1)^{s_{k+1}}L_{s_1+s_2+\cdots+s_k-3k+3-s_{k+1}})
\end{aligned}$$

$$+F_4^{k-1}(-L_{s_1+s_2+\cdots+s_{k+1}-3k-3} - (-1)^{s_{k+1}} L_{s_1+s_2+\cdots+s_k-3k+3-s_{k+1}})$$
$$= (L_3^{k-1} - F_4^{k-1})(-L_{s_1+s_2+\cdots+s_{k+1}-3k-3} - (-1)^{s_{k+1}} L_{s_1+s_2+\cdots+s_k-3k+3-s_{k+1}}) \leq 0.$$

From above, we know that the result is correct.

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A new critical method for twin primes

Fanbei Li

School of Mathematics and Statistics, Inner Mongolia Finance and Economics College,
Hohhot 010051, P.R.China

Abstract For any positive integer $n \geq 3$, if n and $n+2$ both are primes, then we call that n and $n+2$ are twin primes. In this paper, we using the elementary method to study the relationship between the twin primes and some arithmetical function, and give a new critical method for twin primes.

Keywords The Smarandache reciprocal function, critical method for twin primes.

§1. Introduction and result

For any positive integer n, the Smarandache reciprocal function $S_c(n)$ is defined as the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$, and $m+1 \dagger n!$. That is, $S_c(n) = \max\{m : \ y \mid n! \text{ for all } 1 \leq y \leq m, \text{ and } \ m+1 \dagger n!\}$. From the definition of $S_c(n)$ we can easily deduce that the first few values of $S_c(n)$ are:

$$S_c(1) = 1,\ S_c(2) = 2,\ S_c(3) = 3,\ S_c(4) = 4,\ S_c(5) = 6,\ S_c(6) = 6,$$
$$S_c(7) = 10,\ S_c(8) = 10,\ S_c(9) = 10,\ S_c(10) = 10,\ S_c(11) = 12,\ S_c(12) = 12,$$
$$S_c(13) = 16,\ S_c(14) = 16,\ S_5(15) = 16,\ S_c(16) = 16,\ S_c(17) = 18,\ \cdots\cdots .$$

About the elementary properties of $S_c(n)$, many authors had studied it, and obtained a series results, see references [2], [3] and [4]. For example, A.Murthy [2] proved the following conclusion:

If $S_c(n) = x$ and $n \neq 3$, then $x+1$ is the smallest prime greater than n.

Ding Liping [3] proved that for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} S_c(n) = \frac{1}{2} \cdot x^2 + O\left(x^{\frac{19}{12}}\right).$$

On the other hand, Jozsef Sandor [5] introduced another arithmetical function $P(n)$ as follows: $P(n) = \min\{p : \ n \mid p!, \text{ where } p \text{ be a prime}\}$. That is, $P(n)$ denotes the smallest prime p such that $n \mid p!$. In fact function $P(n)$ is a generalization of the Smarandache function $S(n)$. Its some values are: $P(1) = 2$, $P(2) = 2$, $P(3) = 3$, $P(4) = 5$, $P(5) = 5$, $P(6) = 3$, $P(7) = 7$, $P(8) = 5$, $P(9) = 7$, $P(10) = 5$, $P(11) = 11$, $\cdots$. It is easy to prove that for each prime p one has $P(p) = p$, and if n is a square-free number, then $P(n) =$ greatest prime divisor of n. If p be a prime, then the following double inequality is true:

$$2p + 1 \leq P(p^2) \leq 3p - 1 \tag{1}$$

and

$$S(n) \leq P(n) \leq 2S(n) - 1. \tag{2}$$

In reference [6], Li Hailong studied the value distribution properties of $P(n)$, and proved that for any real number $x > 1$, we have the mean value formula

$$\sum_{n \leq x} \left(P(n) - \overline{P}(n)\right)^2 = \frac{2}{3} \cdot \zeta\left(\frac{3}{2}\right) \cdot \frac{x^{\frac{3}{2}}}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\overline{P}(n)$ denotes the largest prime divisor of n, and $\zeta(s)$ is the Riemann zeta-function.

In this paper, we using the elementary method to study the solvability of an equation involving the Smarandache reciprocal function $S_c(n)$ and $P(n)$, and give a new critical method for twin primes. That is, we shall prove the following:

Theorem. For any positive integer $n > 3$, n and $n + 2$ are twin primes if and only if n satisfy the equation

$$S_c(n) = P(n) + 1. \tag{3}$$

§2. Proof of the theorem

In this section, we shall prove our theorem directly. First we prove that if n (> 3) and $n + 2$ both are primes, then n satisfy the equation (3). In fact this time, from A.Murthy [2] we know that $S_c(n) = n + 1$ and $P(n) = n$, so $S_c(n) = P(n) + 1$, and n satisfy the equation (3).

Now we prove that if $n > 3$ satisfy the equation $S_c(n) = P(n) + 1$, then n and $n + 2$ both are primes. We consider n in following three cases:

(A) If $n = q$ be a prime, then $P(n) = P(q) = q$, and $S_c(q) = P(q) + 1 = q + 1$, note that $q > 3$, so from [2] we know that $q + 2$ must be a prime. Thus n and $n + 2$ both are primes.

(B) If $n = q^\alpha$, q be a prime and $\alpha \geq 2$, then from the estimate (2) and the properties of the Smarandache function $S(n)$ we have

$$P\left(q^\alpha\right) \leq 2S\left(q^\alpha\right) - 1 \leq 2\alpha q - 1.$$

On the other hand, from [2] we also have

$$S_c\left(q^\alpha\right) \geq q^\alpha + 2, \text{ if } q \geq 3; \quad \text{and} \quad S_c\left(2^\alpha\right) \geq 2^\alpha + 1.$$

If $S_c\left(q^\alpha\right) = P\left(q^\alpha\right) + 1$, then from the above two estimates we have the inequalities

$$q^\alpha + 3 \leq 2\alpha q \tag{4}$$

and

$$2^\alpha + 2 \leq 4\alpha. \tag{5}$$

It is clear that (4) does not hold if $q \geq 5$ ($q = 3$) and $\alpha \geq 2$ ($\alpha \geq 3$). If $n = 3^2$, then $S_c(9) = 10$, $P(9) = 7$, so we also have $S_c(9) \neq P(9) + 1$.

It is easy to check that the inequality (5) does not hold if $\alpha \geq 4$. $S_c(2) \neq P(2)+1$, $S_c(4) \neq P(4)+1$, $S_c(8) \neq P(8)+1$.

Therefore, if $n = q^{\alpha}$, where q be a prime and $\alpha \geq 2$ be an integer, then n does not satisfy the equation (3).

(C) If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $k \geq 2$ be an integer, p_i $(i = 1, 2, \cdots, k)$ are primes, and $\alpha_i \geq 1$. From the definition of $S_c(n)$ and the inequality (2) we have $S_c(n) \geq n$ and

$$P(n) \leq 2S(n) - 1 = 2 \cdot \max_{1 \leq i \leq k} \{S(p_i^{\alpha_i})\} - 1 \leq 2 \cdot \max_{1 \leq i \leq k} \{\alpha_i p_i\} - 1.$$

So if n satisfy the equation (3), then we have

$$n \leq S_c(n) = P(n) + 1 \leq 2 \cdot S(n) \leq 2 \cdot \max_{1 \leq i \leq k} \{\alpha_i p_i\}.$$

Let $\max\limits_{1 \leq i \leq k} \{\alpha_i p_i\} = \alpha \cdot p$ and $n = p^{\alpha} \cdot n_1$, $n_1 > 1$. Then from the above estimate we have

$$p^{\alpha} \cdot n_1 \leq 2 \cdot \alpha \cdot p. \tag{6}$$

Note that n has at least two prime divisors, so $n_1 \geq 2$, thus (6) does not hold if $p \geq 3$ and $\alpha > 1$. If $p = 2$, then $n_1 \geq 3$. In any case, n does not satisfy the equation (3).

This completes the proof of Theorem.

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$\aleph_0$-spaces and $mssc$-images of relatively compact metric spaces

Nguyen Van Dung

Department of Mathematics, Pedagogical University of Dong Thap,
Dong Thap Province, Vietnam
E-mail: nguyendungtc@yahoo.com

Abstract A space X is an $\aleph_0$-space if and only if X is a sequence-covering, compact-covering $mssc$-image of a relatively compact metric space. This sharpens the main result in [3].

Keywords Relatively compact metric, separable metric, $\aleph_0$-space, $mssc$-mapping, sequence -covering, sequentially-quotient, compact-covering.

§1. Introduction

An investigation of relations between spaces with countable networks and images of separable metric spaces is one of interesting questions on generalized metric spaces. In the past, E. Michael [9] proved that a space is an $\aleph_0$-space if and only if it is a compact-covering image of a separable metric space. Recently, Y. Ge sharpened this result as follows.

Theorem 1.1.([3], Theorem 12) The following are equivalent for a space X.

(1) X is an $\aleph_0$-space.

(2) X is a sequence-covering, compact-covering image of a separable metric space.

(3) X is a sequentially-quotient image of a separable metric space.

Taking this result into account, the following question naturally arises.

Questions 1.2. Can "separable metric", or "image" in Theorem 1.1 be replaced by stronger ones?

In this paper, we affirmatively answer Question1.2 by proving that a space X is an $\aleph_0$-space if and only if X is a sequence-covering, compact-covering $mssc$-image of a relatively compact metric space. This sharpens the main result in [3].

Throughout this paper, all spaces are regular and T_1, $\mathbb{N}$ denotes the set of all natural numbers, $\omega = \mathbb{N} \cup \{0\}$, and a convergent sequence includes its limit point. Let $\mathcal{P}$ be a family of subsets of X. Then $\bigcup \mathcal{P}$, and $\bigcap \mathcal{P}$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$, and the intersection $\bigcap\{P : P \in \mathcal{P}\}$, respectively. A sequence $\{x_n : n \in \omega\}$ converging to x_0 is eventually in $A \subset X$, if $\{x_n : n \geq n_0\} \cup \{x_0\} \subset A$ for some $n_0 \in \mathbb{N}$.

For terms are not defined here, please refer to [2][13].

§2. Main results

Definition 2.1. Let $\mathcal{P}$ be a collection of subsets of a space X.

(1) $\mathcal{P}$ is a pseudobase of X [9], if for every compact subset K and $K \subset U$ with U open in X, there exists $P \in \mathcal{P}$ such that $K \subset P \subset U$.

(2) For each $x \in X$, $\mathcal{P}$ is a network at x in X, if $x \in \bigcap \mathcal{P}$, and if $x \in U$ with U open in X, there exists $P \in \mathcal{P}$ such that $x \in P \subset U$.

(3) $\mathcal{P}$ is a *cs*-network of X [4], if for every convergent sequence S converging to $x \in U$ with U open in X, there exists $P \in \mathcal{P}$ such that S is eventually in $P \subset U$.

(4) $\mathcal{P}$ is a k-network of X [10], if for every compact subset K and $K \subset U$ with U open in X, there exists a finite $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{F} \subset U$.

Definition 2.2. Let X be a space.

(1) X is relatively compact, if $\overline{X}$ is compact.

(2) X is a k-space [2], if $F \subset X$ is closed in X whenever $F \cap K$ is closed in K for every compact subset K of X.

(3) X is an $\aleph_0$-space [9], if X has a countable pseudobase.

Remark 2.3.

(1) It follows from [12, Proposition C] and the regularity of spaces that a space X is an $\aleph_0$-space if and only if X has a countable closed k-network (*cs*-network).

(2) It is easy to see that "compact metric $\Rightarrow$ relatively compact metric $\Rightarrow$ separable metric", and these implications can not be reversed from Example 2.8 and Example 2.9.

Definition 2.4. Let $f : X \longrightarrow Y$ be a mapping.

(1) f is an *mssc*-mapping [6], if X is a subspace of the product space $\prod_{n\in\mathbb{N}} X_n$ of a family $\{X_n : n \in \mathbb{N}\}$ of metric spaces, and for each $y \in Y$, there is a sequence $\{V_{y,n} : n \in \mathbb{N}\}$ of open neighborhoods of y in Y such that each $\overline{p_n(f^{-1}(V_{y,n}))}$ is a compact subset of X_n, where $p_n : \prod_{i\in\mathbb{N}} X_i \longrightarrow X_n$ is the projection.

(2) f is a sequence-covering mapping [11], if for every convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L) = S$.

(3) f is a pseudo-sequence-covering mapping [5], if for every convergent sequence S in Y, there exists a compact subset K of X such that $f(K) = S$.

(4) f is a subsequence-covering mapping [8], if for every convergent sequence S in Y, there exists a compact subset K of X such that $f(K)$ is a subsequence of S.

(5) f is a sequentially-quotient mapping [1], if for every convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L)$ is a subsequence of S.

(6) f is a compact-covering mapping [9], if for every compact subset K of Y, there exists a compact subset L of X such that $f(L) = K$.

Theorem 2.5. The following are equivalent for a space X.

(1) X is an $\aleph_0$-space.

(2) X is a sequence-covering, compact-covering *mssc*-image of a relatively compact metric space.

(3) X is a sequentially-quotient image of a separable metric space.

Proof. (1) $\Rightarrow$ (2). Since X is an $\aleph_0$-space, X has a countable closed cs-network $\mathcal{P}_1$ and a countable closed k-network $\mathcal{P}_2$ by Remark 2.3. Then $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ is a countable closed cs-network and k-network of X. Put $\mathcal{P} = \{P_i : i \in \mathbb{N}\}$, and put $\mathcal{Q}_i = \{P_j : j \leq i\} \cup \{X\} = \{Q_\alpha : \alpha \in A_i\}$, where each A_i is a finite set. Then $X \in \mathcal{Q}_i \subset \mathcal{Q}_{i+1}$. Let every A_i be endowed with the discrete topology. Put

$$M = \Big\{a = (\alpha_i) \in \prod_{i\in\mathbb{N}} A_i : \{Q_{\alpha_i} : i \in \mathbb{N}\} \text{ forms a network at some point } x_a \in X\Big\}.$$

Then M, which is a subspace of the product space $\prod_{i\in\mathbb{N}} A_i$, is a metric space. Since X is T_1 and regular, x_a is unique for each $a \in M$. We define $f : M \longrightarrow X$ by $f(a) = x_a$ for each $a \in M$.

(a) f is onto.

Let $x \in X$. For each $i \in \mathbb{N}$, let $Q_{\alpha_i} = P_i$ if $x \in P_i \in \mathcal{Q}_i$, and otherwise, $Q_{\alpha_i} = X$. Then $\alpha_i \in A_i$ for each $i \in \mathbb{N}$, and $\{Q_{\alpha_i} : i \in \mathbb{N}\}$ forms a network at x in X. Put $a = (\alpha_i)$, then $a \in M$ and $f(a) = x$.

(b) f is continuous.

Let $x = f(a) \in U$ with U open in X and $a \in M$. Put $a = (\alpha_i) \in \prod_{i\in\mathbb{N}} A_i$, where $\{Q_{\alpha_i} : i \in \mathbb{N}\}$ forms a network at x in X. Then there exists $n \in \mathbb{N}$ such that $x \in Q_{\alpha_n} \subset U$. Put $M_a = \{b = (\beta_i) \in M : \beta_n = \alpha_n\}$. Then M_a is an open neighborhood of a in M. For each $b \in M_a$, we get $f(b) \in Q_{\beta_n} = Q_{\alpha_n} \subset U$. It implies that $f(M_a) \subset U$.

(c) f is an *mssc*-mapping.

Let $x \in X$. For each $i \in \mathbb{N}$, put $V_{x,i} = X$. Then $\{V_{x,i} : i \in \mathbb{N}\}$ is a sequence of open neighborhoods of x in X. Since A_i is finite, A_i is compact. Then $\overline{p_i(f^{-1}(V_{x,i}))} = \overline{p_i(f^{-1}(X))} \subset A_i$ is compact. It implies that f is an *mssc*-mapping.

(d) M is relatively compact.

Since A_i is finite, A_i is compact. Then $\prod_{i\in\mathbb{N}} A_i$ is compact, so $\overline{M} \subset \prod_{i\in\mathbb{N}} A_i$ is compact. It implies that M is relatively compact.

(e) f is sequence-covering.

Let $S = \{x_m : m \in \omega\}$ be a convergent sequence converging to x_0 in X. Suppose that U is an open neighborhood of S in X. A family $\mathcal{A}$ of subsets of X has property $cs(S, U)$ if:

(i) $\mathcal{A}$ is finite.

(ii) For each $Q \in \mathcal{A}$, $\emptyset \neq Q \cap S \subset Q \subset U$.

(iii) For each $x_m \in S$, there exists unique $Q_{x_m} \in \mathcal{A}$ such that $x_m \in Q_{x_m}$.

(iv) If $x_0 \in Q \in \mathcal{A}$, then $S \setminus Q$ is finite.

For each $i \in \mathbb{N}$, since $\mathcal{A} = \{X\} \subset \mathcal{Q}_i$ has property $cs(S, X)$ and $\mathcal{Q}_i$ is finite, we can assume that

$$\{\mathcal{A} \subset \mathcal{Q}_i : \mathcal{A} \text{ has property } cs(S, X)\} = \{\mathcal{A}_{ij} : j = n_{i-1} + 1, \ldots, n_i\},$$

where $n_0 = 0$. By this notation, for each $j \in \mathbb{N}$, there is unique $i \in \mathbb{N}$ such that $\mathcal{A}_{ij}$ has property $cs(S, X)$. Then for each $j \in \mathbb{N}$, we can put $\mathcal{A}_{ij} = \{Q_\alpha : \alpha \in E_j\}$, where E_j is a finite subset of A_j.

For each $j \in \mathbb{N}$, $m \in \omega$ and $x_m \in S$, it follows from (iii) that there is unique $\alpha_{jm} \in E_j$ such that $x_m \in Q_{\alpha_{jm}} \in \mathcal{A}_{ij}$. Let $a_m = (\alpha_{jm}) \in \prod_{j\in\mathbb{N}} E_j \subset \prod_{j\in\mathbb{N}} A_j$. Then $\{Q_{\alpha_{jm}} : j \in \mathbb{N}\}$ is a network at x_m in X. In fact, let $x_m \in U$ with U open in X. If $m = 0$, then S is eventually in $Q_{x_0} \subset U$ for some $Q_{x_0} \in \mathcal{Q}$. For each $x \in S \setminus Q_{x_0}$, let $x \in Q_x \subset X \setminus (S \setminus \{x\})$ for some $Q_x \in \mathcal{Q}$. Then $\mathcal{G} = \{Q_{x_0}\} \cup \{Q_x : x \in S \setminus Q_{x_0}\} \subset \mathcal{Q}$ has property $cs(S, X)$. Since $\mathcal{G}$ is finite, $\mathcal{G} \subset \mathcal{Q}_i$ for some $i \in \mathbb{N}$. It implies that $\mathcal{G} = \mathcal{A}_{ij}$ for some $i \in \mathbb{N}$ and some $j \in \{n_{i-1}+1, \ldots, n_i\}$. Since $x_m = x_0 \in Q_{\alpha_{j0}}$, $Q_{\alpha_{j0}} = Q_{x_0}$. Hence $x_m = x_0 \in Q_{\alpha_{j0}} \subset U$. If $m \neq 0$, then $S \setminus \{x_m\}$ is eventually in $Q_{x_0} \subset X \setminus \{x_m\}$ for some $Q_{x_0} \in \mathcal{Q}$. For each $x \in (S \setminus \{x_m\}) \setminus Q_{x_0}$, let $x \in Q_x \subset X \setminus (S \setminus \{x\})$ for some $Q_x \in \mathcal{Q}$, and let $x_m \in Q_{x_m} \subset U \cap (X \setminus (S \setminus \{x_m\}))$ for some $Q_{x_m} \in \mathcal{Q}$. Then $\mathcal{H} = \{Q_{x_0}\} \cup \{Q_{x_m}\} \cup \{Q_x : x \in (S \setminus \{x_m\}) \setminus Q_{x_0}\}$ has property $cs(S, X)$. Since $\mathcal{H}$ is finite, $\mathcal{H} \subset \mathcal{Q}_i$ for some $i \in \mathbb{N}$. It implies that $\mathcal{H} = \mathcal{A}_{ij}$ for some $i \in \mathbb{N}$ and some $j \in \{n_{i-1} + 1, \ldots, n_i\}$. Since $x_m \in Q_{\alpha_{jm}}$, $Q_{\alpha_{jm}} = Q_{x_m}$. Hence $x_m \in Q_{\alpha_{jm}} \subset U$.

By the above, for each $m \in \omega$, we get $a_m = (\alpha_{jm}) \in M$ satisfying $f(a_m) = x_m$. For each $j \in \mathbb{N}$, since families $\mathcal{H}$ and $\mathcal{G}$ are finite, there exists $m(j) \in \mathbb{N}$ such that $\alpha_{jm} = \alpha_{j0}$ if $m \geq m(j)$. Hence the sequence $\{\alpha_{jm} : m \in \mathbb{N}\}$ converges to α_{j0} in A_j. Thus, the sequence $\{a_m : m \in \mathbb{N}\}$ converges to a_0 in M. Put $L = \{a_m : m \in \omega\}$, then L is a convergent sequence in M and $f(L) = S$. This shows that f is sequence-covering.

(f) f is compact-covering.

Let K be a compact subset of X. Suppose that V is an open neighborhood of K in X. A family $\mathcal{B}$ of subsets of X has property $k(K, V)$ if:

(i) $\mathcal{B}$ is finite.

(ii) $Q \cap K \neq \emptyset$ for each $Q \in \mathcal{B}$.

(iii) $K \subset \bigcup \mathcal{B} \subset V$.

For each $i \in \mathbb{N}$, since $\mathcal{B} = \{X\} \subset \mathcal{Q}_i$ has property $k(K, X)$ and $\mathcal{Q}_i$ is finite, we can assume that

$$\{\mathcal{B} \subset \mathcal{Q}_i : \mathcal{B} \text{ has property } k(K, X)\} = \{\mathcal{B}_{ij} : j = n_{i-1} + 1, \ldots, n_i\},$$

where $n_0 = 0$. By this notation, for each $j \in \mathbb{N}$, there is unique $i \in \mathbb{N}$ such that $\mathcal{B}_{ij}$ has property $k(K, X)$. Then for each $j \in \mathbb{N}$, we can put $\mathcal{B}_{ij} = \{Q_\alpha : \alpha \in F_j\}$, where F_j is a finite subset of A_j.

Put $L = \{a = (\alpha_i) \in \prod_{i\in\mathbb{N}} F_i : \bigcap_{i\in\mathbb{N}} (K \cap Q_{\alpha_i}) \neq \emptyset\}$. We shall prove that L is a compact subset of M satisfying that $f(L) = K$, hence f is compact-covering, by the following facts (i), (ii), and (iii).

(i) L is compact.

Since $L \subset \prod_{i\in\mathbb{N}} F_i$ and $\prod_{i\in\mathbb{N}} F_i$ is compact, we only need to prove that L is closed in $\prod_{i\in\mathbb{N}} F_i$. Let $a = (\alpha_i) \in \prod_{i\in\mathbb{N}} F_i \setminus L$. Then $\bigcap_{i\in\mathbb{N}} (K \cap Q_{\alpha_i}) = \emptyset$. Since $K \cap Q_{\alpha_i}$ is closed in K for every $i \in \mathbb{N}$ and K is compact, there exists $i_0 \in \mathbb{N}$ such that $\bigcap_{i\leq i_0} (K \cap Q_{\alpha_i}) = \emptyset$. Put $W = \{b = (\beta_i) \in \prod_{i\in\mathbb{N}} F_i : \beta_i = \alpha_i \text{ if } i \leq i_0\}$. Then W is an open neighborhood of a in $\prod_{i\in\mathbb{N}} F_i$ and $W \cap L = \emptyset$. If not, there

exists $b = (\beta_i) \in W \cap L$. Since $b \in L$, $\bigcap_{i\in\mathbb{N}}(K \cap Q_{\beta_i}) \neq \emptyset$, hence $\bigcap_{i\leq i_0}(K \cap Q_{\beta_i}) \neq \emptyset$. Since $b \in W$, $\bigcap_{i\leq i_0}(K \cap Q_{\alpha_i}) = \bigcap_{i\leq i_0}(K \cap Q_{\beta_i}) \neq \emptyset$. This is a contradiction of the fact that $\bigcap_{i\leq i_0}(K \cap Q_{\alpha_i}) = \emptyset$.

(ii) $L \subset M$ and $f(L) \subset K$.

Let $a = (\alpha_i) \in L$, then $a \in \prod_{i\in\mathbb{N}} F_i$ and $\bigcap_{i\in\mathbb{N}}(K \cap Q_{\alpha_i}) \neq \emptyset$. Pick $x \in \bigcap_{i\in\mathbb{N}}(K \cap Q_{\alpha_i})$. If $\{Q_{\alpha_i} : i \in \mathbb{N}\}$ is a network at x in X, then $a \in M$ and $f(a) = x$, hence $L \subset M$ and $f(L) \subset K$. So we only need to prove that $\{Q_{\alpha_i} : i \in \mathbb{N}\}$ is a network at x in X. Let V be an open neighborhood of x in X. There exist an open subset W of K such that $x \in W$, and compact subsets $cl_K(W)$ and $K \setminus W$ such that $cl_K(W) \subset V$ and $K \setminus W \subset X \setminus \{x\}$, where $cl_K(W)$ is the closure of W in K. Since $\mathcal{Q}$ is a k-network of X, there exist finite families $\mathcal{Q}_1 \subset \mathcal{Q}$ and $\mathcal{Q}_2 \subset \mathcal{Q}$ such that $cl_K(W) \subset \bigcup \mathcal{Q}_1 \subset V$ and $K \setminus W \subset \bigcup \mathcal{Q}_2 \subset X \setminus \{x\}$. We may assume that $Q \cap K \neq \emptyset$ for each $Q \in \mathcal{Q}_1 \cup \mathcal{Q}_2$. Put $\mathcal{L} = \mathcal{Q}_1 \cup \mathcal{Q}_2$, then $\mathcal{L}$ has property $k(K, X)$. It implies that $\mathcal{L} = \mathcal{B}_{ij}$ for some $i \in \mathbb{N}$ and some $j \in \{n_{i-1} + 1, \ldots, n_i\}$. Since $x \in Q_{\alpha_j} \in \mathcal{B}_{ij}, Q_{\alpha_j} \in \mathcal{Q}_1$, thus $Q_{\alpha_j} \subset V$. This prove that $\{Q_{\alpha_j} : j \in \mathbb{N}\}$ is a network at x in X.

(iii) $K \subset f(L)$.

Let $x \in K$. For each $i \in \mathbb{N}$, there exists $\alpha_i \in F_i$ such that $x \in Q_{\alpha_i}$. Put $a = (\alpha_i)$, then $a \in L$. Furthermore, $f(a) = x$ as in the proof of (ii). So $K \subset f(L)$.

(2) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (2). It follows from [3, Lemma 11].

Corollary 2.6. The following are equivalent for a space X.

(1) X is a k-and-$\aleph_0$-space.

(2) X is a sequence-covering, compact-covering, quotient *mssc*-image of a relatively compact metric space.

(3) X is a quotient *mssc*-image of a separable metric space.

Remark 2.7. It follows from Remark 2.3, Definition 2.4, and [3, Lemma 10] that "sequentially-quotient", "relatively compact metric", "image" in the above results can be replaced by "sequence-covering" ("compact-covering", "pseudo-sequence-covering", "subsequence-covering"), "separable metric", "*mssc*-image", respectively. Then Theorem 2.5 sharpens the main result in [3].

Finally, we give examples to illustrate the above results.

Let $\mathbb{R}$ and $\mathbb{Q}$ be the set of all real numbers and rational numbers endowed with the usual topology, respectively.

Example 2.8. A relatively compact metric space is not compact.

Proof. Let $M = (0, 1) \subset \mathbb{R}$. Then M is a relatively compact metric space, which is not compact.

Example 2.9. A separable metric space is not relatively compact.

Proof. Recall that $\mathbb{Q}$ is a separable metric space. Since $\overline{\mathbb{Q}} = \mathbb{R}$ and $\mathbb{R}$ is not compact, $\mathbb{Q}$ is not relatively compact.

Example 2.10. A sequence-covering, compact-covering mapping from a separable metric space is not an *mssc*-mapping.

Proof. Recall that $\mathbb{Q}$ is a non-locally compact, separable metric space. Put $M = \mathbb{Q} \times \{0\} \times \cdots \times \{0\} \cdots \subset \prod_{i\in\mathbb{N}} X_i$, where $X_i = \mathbb{Q}$ for each $i \in \mathbb{N}$. It is clear that M is a separable metric

space. Define $f : M \longrightarrow \mathbb{Q}$ by $f(x, 0, \dots) = x$ for each $x \in \mathbb{Q}$. Then f is a sequence-covering, compact-covering mapping from a separable metric space. If f is an $mssc$-mapping, then, for each $x \in \mathbb{Q}$, there exists a sequence $\{V_{x,i} : i \in \mathbb{N}\}$ of open neighborhoods of x in $\mathbb{Q}$ such that each $\overline{p_i(f^{-1}(V_{x,i}))}$ is a compact subspace of X_i. Thus, $\overline{p_1(f^{-1}(V_{x,1}))}$ is a compact subset of $\mathbb{Q}$, so $\mathbb{Q}$ is a locally compact space. It is a contradiction. Hence f is not an $mssc$-mapping.

Example 2.11. An $\aleph_0$-space is not any image of a compact metric space. It implies that "relatively compact metric" in the above results can not be replaced by "compact metric".

Proof. Recall that $\mathbb{R}$ is an $\aleph_0$-space. Since $\mathbb{R}$ is not compact, $\mathbb{R}$ is not any image of a compact metric space.

Example 2.12. An $\aleph_0$-space is not any sequence-covering, compact-covering compact image of a metric space. It implies that "$mssc$-image" in the above results can not be replaced by "compact image".

Proof. Recall that S_ω is a Fréchet and $\aleph_0$-space (see [7], Example 1.8.7, for example). It follows from [13, Remark 4] that S_ω is not any quotient compact image of a metric space. Then X is not any sequence-covering, compact-covering compact image of a metric space.

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On two conjectures by K. Kashihara on prime numbers

József Sándor

Babeş–Bolyai University of Cluj, Romania

E-mail: jjsandor@hotmail.com

Abstract We settle two conjectures posed by K. Kashihara in his book [1]. The first conjecture states that $\prod_{i=1}^{n}\left(1-\frac{1}{p_i}\right) < \frac{1}{p_{n+1}-p_n}$ for all n; while the second one that the sequence of general term $\sum_{i=1}^{n} p_i^2 \Big/ \left(\sum_{i=1}^{n} p_i\right)^2$ is convergent. Here p_n denotes the nth prime. We will prove that the first conjecture is false for sufficiently large n. The second conjecture is true, the limit being zero.

Keywords prime numbers, estimates on primes, convergence of sequences.

§1. Introduction

Let p_n denote the nth prime number. In his book [2], K. Kashihara posed several conjectures and open problems. On page 45 it is conjectured the following inequality:

$$p_{n+1}-p_n < \prod_{i=1}^{n}\frac{1}{1-\frac{1}{p_i}}, \quad (n=1,2,\ldots) \tag{1}$$

A numerical evidence suggests that this inequality may be true for all values of n. However, as we will see, for large values of n, relation (1) cannot hold.

Another conjecture (see page 46) states that the sequence (x_n) of general term

$$x_n = \frac{\sum_{i=1}^{n} p_i^2}{\left(\sum_{i=1}^{n} p_i\right)^2} \quad (n \geq 1) \tag{2}$$

is convergent, having a limit between 1,4 and 1,5. Though this sequence is indeed convergent, we will see that its limit is $\rho = 0$.

§2. Proof of the theorem

An old theorem of F. Mertens (see e.g. [3], p.259) states that

$$\prod_{p\leq x}\left(1-\frac{1}{p}\right) \sim \frac{c}{\log x} \text{ as } x \to \infty, \tag{3}$$

where $c = e^{-\gamma}$ (e and γ being the two Euler constants). Inequality (1) can be written also as

$$\prod_{p \le p_n} \left(1 - \frac{1}{p}\right) < \frac{1}{p_{n+1} - p_n}. \tag{4}$$

Since the first term of (4) is $\sim \dfrac{c}{\log p_n}$, if (4) would be true, then for all $\varepsilon > 0$ (fixed) and $n \ge n_0$ we would obtain that $\dfrac{1}{p_{n+1} - p_n} > \prod\limits_{p \le p_n} \left(1 - \dfrac{1}{p}\right) > \dfrac{c - \varepsilon}{\log p_n}$. Let $\varepsilon = \dfrac{c}{2} > 0$. Then $\dfrac{c}{2} \cdot \dfrac{1}{\log p_n} < \dfrac{1}{p_{n+1} - p_n}$, so $b_n = \dfrac{p_{n+1} - p_n}{\log p_n} < \dfrac{2}{c} = K$. This means that the sequence of general term (b_n) is bounded above. On the other hand, a well-known theorem by E. Westzynthius (see [3], p. 256) states that $\lim\limits_{n \to \infty} \sup b_n = +\infty$, i.e. the sequence (b_n) is unbounded. This finishes the proof of the first part.

For the proof of convergence of (x_n) given by (2), we shall apply the result

$$\sum_{p \le x} p^{\alpha} \sim \frac{x^{1+\alpha}}{(1+\alpha)\log x} \text{ as } x \to \infty \ (\alpha \ge 0) \tag{5}$$

due to T. Salát and S. Znám (see [3], p. 257). We note that for $\alpha = 1$, relation (5) was discovered first by E. Landau. Now, let $\alpha = 1$, resp. $\alpha = 2$ in (5), we can write:

$$\sum_{p \le p_n} p \sim \frac{p_n^2}{2 \log p_n} \text{ as } n \to \infty; \tag{6}$$

and

$$\sum_{p \le p_n} p^2 \sim \frac{p_n^3}{3 \log p_n} \text{ as } n \to \infty. \tag{7}$$

Thus, $x_n = \left[\left(\sum\limits_{p \le p_n} p^2 \right) \cdot \dfrac{3 \log p_n}{p_n^3} \cdot \dfrac{p_n^4}{\left(\sum\limits_{p \le p_n} p \right)^2 \cdot 4 \log^2 p_n} \right] \cdot \dfrac{4}{3} \cdot \dfrac{\log p_n}{p_n}$.

By (6) and (7), the limit of term $[\ldots]$ is 1. Since $\dfrac{4}{3} \cdot \dfrac{\log p_n}{p_n} \to 0$, we get $\lim\limits_{n \to \infty} x_n = 0$. This finishes the proof of the second part.

Remarks.

1) An extension of (5) is due to M. Kalecki [1]:

Let $f : (0, +\infty) \to \mathbb{R}$ be an arbitrary function having the following properties:

a) $f(x) > 0$; b) $f(x)$ is a non-decreasing function; c) for each $n > 0$, $\varphi(n) = \lim\limits_{x \to \infty} \dfrac{f(nx)}{f(x)}$ exists.

Put $s = \log \varphi(e)$. Then

$$\sum_{p \le x} f(p) \sim \frac{f(x) \cdot x}{\log x} \cdot \frac{1}{s+1} \text{ as } x \to \infty. \tag{8}$$

For $f(x) = x^\alpha$ ($\alpha \geq 0$) we get $\varphi(n) = n^\alpha$, so $s = \alpha$ and relation (5) is reobtained. We note that for $\alpha = 0$, relation (5) implies the "prime number theorem" ([3])

$$\pi(x) \sim \frac{x}{\log x} \text{ as } x \to \infty,$$

where $\pi(x) = \sum_{p \leq x} 1 =$ number of primes $\leq x$.

2) By letting $f(x) = (g(x))^\alpha$, where g satisfies conditions $a) - c)$ a general sequence of terms $x_n = \sum_{i=1}^{n} (g(p_i))^\alpha \Big/ \left(\sum_{i=1}^{n} g(p_i) \right)^\alpha$ may be studied (via (8)) in a similar manner. We omit the details.

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Some properties of the lattice cubic

A.A.K. Majumdar

Beppu-shi 875-8577, Oita-ken, Japan
majumdar@apu.ac.jp aakmajumdar@gmail.com

Abstract This paper considers some of the properties of the lattice cubic $y = x^3$. We show that the area of any triangle inscribed in the lattice cubic is integer-valued. We find that a geometrical problem leads to the Diophantine equation $4p^2 = (2q + r)^2 + r^2$. We study the implications of the solutions of this Diophantine equation.

Keywords Lattice point, lattice triangle, lattice cubic, Diophantine equation

§1. Introduction

On a two-dimensional coordinate plane, a point is represented by an ordered pair of numbers (x, y). Of particular interest is a lattice point, defined below.

Definition 1.1. The point (x, y) on the xy-plane is called a lattice point if both x and y are integers.

Throughout this paper, we shall denote by $\mathbb{Z}$ the set of all integers, and by $\mathbb{Z}^+$ the set of all positive integers, and by $\mathbf{N}$ the set of all positive integers including 0.

Definition 1.2. A lattice triangle on the xy-plane is one whose vertices are all integers. A lattice triangle is called Heronian if its sides as well as the area are all positive integers.

Definition 1.3. The lattice cubic consists of all points (x, y) on the parabola $y = x^3$ such that $x, y \in \mathbb{Z}$.

In this paper, we consider some of the properties of the lattice cubic. We show that a geometrical problem in the lattice cubic gives rise to a Diophantine equation. Some of the properties were studied by Majumdar [1]. Here, we particularly focus on the nature of the solution of the Diophantine equation.

§2. The lattice cubic $y = x^3$

A lattice triangle $\triangle PQR$, inscribed in the lattice cubic $y = x^3$, can be described by its vertices $P(p, p^3)$, $Q(q, q^3)$ and $R(r, r^3)$, where $p, q, r \in \mathbb{Z}$. Without loss of generality, we may assume that $p < q < r$.

Now, the slope of the line PQ is $\dfrac{q^3 - p^3}{q - p} = q^2 + pq + p^2$. Since

$$q^2 + pq + p^2 = \frac{1}{4}\left[(2q + p)^2 + 3p^2\right]$$

for any $p \neq 0$ and $q \neq 0$, it follows that the quadratic form $q^2 + pq + p^2$ is positive definite. Thus,

$$(q^2 + pq + p^2)(r^2 + qr + q^2) > 0$$

for any $p \neq 0$ and $q \neq 0$. But $r^2 + qr + q^2$ is the slope of the line QR. It thus follows that no right-angled triangle can be inscribed in the lattice cubic $y = x^3$.
Again, since

$$PQ = \sqrt{(q-p)^2 + (q^3 - p^3)^2} = (q-p)\sqrt{1 + (q^2 + pq + p^2)^2},$$

and since $1+(q^2+pq+p^2)^2$ can not be a perfect square, it follows that no Heronian triangle can be inscribed in the lattice cubic $y = x^3$. However, the area of the triangle $\triangle PQR$ is integer-valued, as the following lemma shows.

Lemma 2.1. The area of the lattice triangle with vertices at the lattice points $P(p, p^3)$, $Q(q, q^3)$ and $R(r, r^3)$ with $p < q < r$ on the lattice cubic $y = x^3$, is

$$\triangle(p, q, r) \equiv \frac{1}{2}(q-p)(r-q)(r-p)|p+q+r|. \tag{1}$$

Proof. The area of the triangle $PQR$is the absolute value of

$$\frac{1}{2}\begin{vmatrix} 1 & p & p^3 \\ 1 & q & q^3 \\ 1 & r & r^3 \end{vmatrix}.$$

Now,

$$\begin{vmatrix} 1 & p & p^3 \\ 1 & q & q^3 \\ 1 & r & r^3 \end{vmatrix} \begin{vmatrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \end{vmatrix} = \begin{vmatrix} 1 & p & p^3 \\ 0 & q-p & (q-p)(q^2+pq+p^2) \\ 0 & r-q & (r-q)(r^2+qr+q^2) \end{vmatrix}$$

$$= (q-p)(r-q)\begin{vmatrix} 1 & q^2+pq+p^2 \\ 1 & r^2+qr+q^2 \end{vmatrix}$$

which gives the desired result.

In (1), let

$$m = q - p, \quad n = r - q. \tag{2}$$

Then, from Lemma 2.1, the area of the triangle PQR can be written as

$$\triangle(p, q, r) = \frac{1}{2}mn(m+n)|3p + 2m + n|; \quad p \in \mathbb{Z}; \quad m, n \in \mathbb{Z}^+ \tag{3}$$

In [1], we proved the following result. Here, we use a different approach, taking into account the possible forms of the integers involved. This form would be helpful in the analysis of the possible values of the inscribed triangles later.

Lemma 2.2. The area of the lattice triangle with vertices at $P(p, p^3), Q(q, q^3)$ and $R(r, r^3)$ is

$$\triangle(p, q, r) = 3\ell$$

for some $\ell \in \mathbb{Z}$.

Proof. By (3), the area of the triangle PQR can be expressed as

$$\triangle(p,q,r) = \frac{1}{2}mn(m+n)|3p+2m+n|; \quad p \in \mathbb{Z}; \quad m,n \in \mathbb{Z}^+ \tag{4}$$

We consider the following nine possibilities that may arise :

(1) $m = 3k_1+1, \quad n = 3k_2+1$ for some integers $k_1, k_2 \geq 0$. In this case,

$$\triangle(p,q,r) = \frac{3}{2}(3k_1+1)(3k_2+1)[3(k_1+k_2)+2]|p+2k_1+k_2+1|.$$

(2) $m = 3k_1+1, \quad n = 3k_2+2$ for some integers $k_1, k_2 \geq 0$. In this case,

$$\triangle(p,q,r) = \frac{3}{2}(3k_1+1)(3k_2+2)(k_1+k_2+1)|3(p+2k_1+k_2)+4|.$$

(3) $m = 3k_1+2, \quad n = 3k_2+1$ for some integers $k_1, k_2 \geq 0$. In this case,

$$\triangle(p,q,r) = \frac{3}{2}(3k_1+2)(3k_2+1)(k_1+k_2+1)|3(p+2k_1+k_2)+5|.$$

(4) $m = 3k_1+2, \quad n = 3k_2+2$ for some integers $k_1, k_2 \geq 0$. In this case,

$$\triangle(p,q,r) = \frac{3}{2}(3k_1+2)(3k_2+2)[3(k_1+k_2)+4]|p+2k_1+k_2+2|.$$

(5) $m = 3k_1, \quad n = 3k_2+1$ for some integers $k_1 \geq 1, k_2 \geq 0$. In this case,

$$\triangle(p,q,r) = \frac{3}{2}k_1(3k_2+1)[3(k_1+k_2)+4]|3(p+2k_1+k_2)+1|.$$

(6) $m = 3k_1+1, \quad n = 3k_2$ for some integers $k_1 \geq 1, k_2 \geq 0$. In this case,

$$\triangle(p,q,r) = \frac{3}{2}(3k_1+1)k_2[3(k_1+k_2)+1]|3(p+2k_1+k_2)+2|.$$

(7) $m = 3k_1, \quad n = 3k_2+2$ for some integers $k_1 \geq 1, k_2 \geq 0$. In this case,

$$\triangle(p,q,r) = \frac{3}{2}k_1(3k_2+2)[3(k_1+k_2)+2]|3(p+2k_1+k_2)+2|.$$

(8) $m = 3k_1+2, \quad n = 3k_2$ for some integers $k_1 \geq 0, k_2 \geq 1$. In this case,

$$\triangle(p,q,r) = \frac{3}{2}(3k_1+2)k_2[3(k_1+k_2)+2]|3(p+2k_1+k_2)+4|.$$

(9) $m = 3k_1, \quad n = 3k_2$ for some integers $k_1, k_2 \geq 1$.

In this case,

$$\triangle(p,q,r) = \frac{3^4}{2}k_1k_2(k_1+k_2)|p+2k_1+k_2|.$$

Thus, in all the cases, $\triangle(p,q,r)$ is a multiple of 3, establishing the lemma.

By symmetry, if $\triangle(p,q,r) = 3\ell$ then $\triangle(-p,-q,-r) = 3\ell$ that is, the area of the triangle with vertices $P'(-p,-p^3)$, $Q'(-q,-q^3)$ and $R(-r,-r^3)$is also 3ℓ. Now, given any integer $\ell \geq 1$, is it always possible to find a triangle with area ℓ? The answer is yes : For example, in Case (1) in the proof of Lemma 2.2, putting $p = 0, 1, 2, \ldots$ successively, we get the triangles of areas $3, 6, 9, \ldots$. The next question is

Question 1. Is it possible to find some formula for all the lattice triangles inscribed in the lattice cubic $y = x^3$, each having the area 3ℓ for any fixed integer $\ell \geq 1$?

In connection with Question 1 above, we observe the following facts from the proof of Lemma 2.2 :

Case (1) : For $k_1 = 0 = k_2, \triangle(p, q, r) = 3|p + 1|$.

Case (2) : For $k_1 = 0 = k_2, \triangle(p, q, r) = 3|3p + 4|$.

Case (3) : For $k_1 = 0 = k_2, \triangle(p, q, r) = 3|3p + 5|$.

Case (5) : For $k_1 = 1, k_2 = 0, \triangle(p, q, r) = 6|3p + 7|$.

Case (6) : For $k_1 = 0, k_2 = 1, \triangle(p, q, r) = 6|3p + 5|$.

(1) The minimum-area triangles, each of area 3, can be obtained from Case (1) with $p = 0$, and Case (2) with $p = -1$. Thus, we get the triangles $\triangle(0, 1, 2)$ and $\triangle(-2, -1, 0)$ (in the notation of (1) and (2)), as well as the triangles $\triangle(-1, 0, 2)$ and $\triangle(-2, 0, 1)$. Note that, Case (3) with $p = -2$ does not give any different triangle.

Hence, there are, in total, four triangles, each of area 3.

(2) To find the triangles, each of area 6, we put p = 1 in Case (1), $p = -2$ in Case (2), and $p = -2$ in Case (5). Corresponding to these values, the triangles are $\triangle(1, 2, 3)$ (and hence, also the triangle $\triangle(-3, -2, -1)$), $\triangle(-2, -1, 1)$, (and the triangle $\triangle(-1, 1, 2)$), and $\triangle(-2, 1, 2)$ (and the triangle $\triangle(-2, -1, 2)$). No further triangles are obtained from Case (3) with p = - 1 and Case (6) with p = - 2.

Thus, there are six triangles, each with area 6.

(3) There are only two triangles, each of area 9. These can be obtained from Case (1) with $p = 2$. Thus, the desired triangles are $\triangle(2, 3, 4)$ and $\triangle(-4, -3, -2)$.

(4) To find the triangles, each of area 12, note that Case (1) with $p = 3$ (or, $p = -5$), Case (2) with $p = 0$ (Case (3) with $p = -3$ gives the same triangles), and Case (5) with $p = -3$ (or, Case (6) with $p = -1$), give such triangles. The triangles of interest are $\triangle(3, 4, 5)$ (and $\triangle(-5, -4, -3)$), $\triangle(0, 1, 3)$ (and $\triangle(-3, -1, 0)$) and $\triangle(-1, 0, 3)$ (and $\triangle(-3, 0, 1)$).

Thus, there are six triangles, each of area 12.

(5) To find the triangles, each of area 15, we put $p = 4$ (or, $p = -6$) in Case (1), $p = -3$ in Case (2) (Case (3) with $p = 0$ gives the same triangles), and $p = -3, k_1 = 1, k_2 = 0$ in Case (7) (Case (8) with $p = -2, k_1 = 0, k_2 = 1$ gives the same triangles). Then, we get the triangles $\triangle(4, 5, 6)$(and $\triangle(-6, -5, -4)$), $\triangle(0, 2, 3)$ (and $\triangle(-3, -2, 0)$) and $\triangle(-2, 0, 3)$ (and $\triangle(-3, 0, 2)$). Thus, there are six triangles, each of area 15.

Given any two distinct points $P(p, p^3)$ and $Q(q, q^3)$ with $p \neq 0, q \neq 0$ and $q \neq -p$, on the lattice cubic $y = x^3$, we can always find a line parallel to PQ and intersecting the lattice cubic, namely, the line joining the points $P'(-p, -p^3)$ and $Q'(-q, -q^3)$. But what happens, if we choose the line PO, passing through the point $P'(-p, -p^3)$ on the lattice cubic and the origin $O(0, 0)$? More precisely, let $P(-p, -p^3))$, $p \neq 0$, be any point on the lattice cubic $y = x^3$. Then, the line PO would intersect the lattice cubic at the second point $P'(-p, -p^3)$. The question is : Is there any line parallel to POP' and intersecting the lattice cubic? To answer this question, let $Q(q, q^3)$ and $R(r, r^3)$ be two distinct points on the lattice cubic $y = x^3$ such that QR is parallel to POP' . Then, we have the following result.

Lemma 2.3. The line QR (where Q and R are the lattice points $Q(q, q^3)$, $R(r, r^3)$ on

the lattice cubic $y = x^3$) is parallel to the line $POP^{'}$ (where P and $P^{'}$ are the points $P(p, p^3)$, $P^{'}(-p, -p^3)$) if and only if p, q and r satisfy the Diophantine equation

$$p^2 = q^2 + qr + r^2. \tag{5}$$

Proof. The slope of the line $POP^{'}$ is p^2, and that of the line QR is $q^2 + qr + r^2$. Thus, these two lines are parallel if and only if $p^2 = q^2 + qr + r^2$.

By inspection, we have the following solutions of the Diophantine equation (5) :

(1) $q = \pm p, r = \mp p$;

(2) $q = \pm p, r = 0$;

(3) $q = 0, r = \pm p$.

These are the trivial solutions. To find the points $Q(q, q^3)$ and $R(r, r^3)$, we have to look for the non-trivial solutions of the Diophantine equation (5). Moreover, we may assume, without loss of generality, that $p > 0$; also, we may assume that $q > 0$ and $r > 0$, because, by symmetry, if QR is parallel to $POP^{'}$, then $Q^{'}R^{'}$ is also parallel to $POP^{'}$, where $Q^{'}(-q, -q^3)$ and $R^{'}(-r, -r^3)$. Thus, the problem of finding the line parallel to $POP^{'}$ reduces to the problem of finding positive non-trivial solutions of the Diophantine equation (5). We now observe the following facts :

(1) if $(p_0, q_0, r_0)$is a solution of the Diophantine equation (5), so is (p_0, r_0, q_0);

(2) if (p_0, q_0, r_0) is a solution of (5), so is (kp_0, kq_0, kr_0) for any $k \in \mathbb{Z}^+$.

Note that the solutions (p_0, q_0, r_0) and (p_0, r_0, q_0) are the same. By virtue of the second observation, it is sufficient to look for solutions of the Diophantine equation (5) for primes p only.

Writing the Diophantine equation (5) in the form,

$$4p^2 = (2q + r)^2 + 3r^2. \tag{6}$$

We searched for the solution of the Diophantine equation (6) for $1 \leqslant p \leqslant 100$. The following table summarizes our findings.

Table 2.1 : Solutions of $4p^2 = (2q + r)^2 + 3r^2$ **,**$1 \leqslant p \leqslant 100$

p	q	r		p	q	r		p	q	r
7	3	5		49	16	39		91	11	85
13	7	8			21	35			19	80
19	5	16		61	9	56			39	65
31	11	24		67	32	45			49	56
37	7	33		73	17	63		97	55	57
43	13	35		79	40	51				

The above table shows that, there are solutions of the Diophantine equation (6) only for the primes $p = 7, 13, 19, 31, 37, 43, 61, 67, 73, 79$ and 97 (and their multiples) on the range $1 \leqslant p \leqslant 100$, and in each case, there is only one solution. Thus, corresponding to each of these

values, there are only two lines parallel to the line $POP^{'}$; thus, for example, there are only two lines, namely, the line passing through the points $Q(q, q^3) = Q(3, 3^3)$ and $R(r, r^3) = R(5, 5^3)$, and (by symmetry) the line passing though the points $(-3, -3^3)$ and $(-5, -5^3)$, which are parallel to the line through the origin $0(0, 0)$ and the point $(7, 7^3)$. However, there are four lines parallel to the line through $(49, 49^3)$ and 0(0,0); the four lines are those passing through the points $(16, 16^3)$ and $(39, 39^3)$ (and the line through $(-16, -16^3)$ and $(-39, -39^3)$, together with the line passing through the points $(21, 21^3)$ and $(35, 35^3)$ (as well as the line through $(-21, -21^3)$ and $(-35, -35^3)$) And there are eight lines, each of which is parallel to the line through the origin 0 (0, 0) and the point $(91, 91^3)$: The line through the points (11, 11^3) and (85, 85^3) (and that through (- 11, - 11^3) and (- 85, - 85^3)), the line through the points (19, 19^3) and (80, 80^3) (and the line through (- 19, - 19^3) and (- 80, - 80^3)), the line through the points (39, 39^3) and (65, 65^3) (together with the line through (- 39, - 39^3) and (- 65, - 65^3)), and the line through the points (49, 49^3) and (56, 56^3) (as well as the line passing through the points (- 49, - 49^3) and (- 56, - 56^3)).

Our second question is

Question 2. Is it possible to determine a formula that would give all the lines parallel to the line through the origin $0(0, 0)$ and the point $P(p, p^3)$, $p \neq 0$, on the lattice cubic $y = x^3$?

Now, we consider the problem of finding all lattice triangles, inscribed in the lattice cubic $y = x^3$, whose areas are perfect squares, that is, the triangles such that

$$\triangle(p, q, r) = a^{'2}$$

for some integer $a^{'} \geqslant 1$. Now, since $3|\triangle(p, q, r)$ (by Lemma 2.2), it follows that $3|a^{'}$. Thus, the triangles whose areas are perfect squares must be such that

$$\triangle(p, q, r) = (3a)^2$$

for some integer $a \geq 1$.

It is always possible to find a lattice triangle inscribed in the lattice cubic $y = x^3$, whose area is a perfect square. Recall that, triangles with area $(3a)^2$ occur in pairs, that is, if the area of the triangle with vertices at the points $P(p, p^3)$, $Q(q, q^3)$ and $R(r, r^3)$ is $(3a)^2$, then the area of the triangle with vertices at $P^{'}(-p, -p^3)$, $Q^{'}(-q, -q^3)$ and $R^{'}(-r, -r^3)$ is also $(3a)^2$. We already found two triangles, each with area 9.

In fact, we can prove a more general result.

Lemma 2.4. There exists an infinite number of lattice triangles inscribed in the lattice cubic $y = x^3$, each with an area which is a perfect square.

Proof. We prove the lemma by actually constructing a family of lattice triangles, each having an area which is a perfect square. To do so, we proceed as follows : In (2.3), let $m = n$. Then,

$$\triangle(p, q, r) = 3m^3|p + m|. \tag{7}$$

To make (7) a perfect square, let

$$p = (3n^2 - 1)m; m, n \in \mathbb{Z}^+.$$

The resulting triangle is $\triangle\big((3n^2-1)m, 3n^2m, (3n^2+1)m\big)$.
Hence, $\triangle\big((3n^2-1)m, 3n^2m, (3n^2+1)m\big)$; $m, n \in \mathbb{Z}^+$, is the desired family, with the area $(3m^2n)^2$.

We now pose the following question

Question 3. Is it possible to characterize all the triangles inscribed in the lattice cubic $y = x^3$, whose areas are perfect squares?

Next, we consider the problem of finding all inscribed lattice triangles whose areas are cubes of natural numbers. By Lemma 2.2, the areas of such triangles must be of the form

$$\triangle(p, q, r) = (3a)^3$$

for some integer $a \geq 1$. We can prove the following result.

Lemma 2.5. There exists an infinite number of lattice triangles inscribed in the lattice cubic $y = x^3$, each with an area which is cube of a natural number.

Proof. We proceed on the same line of proof as of Lemma 2.4. In (7), letting

$$p = 3^2n^3 - m; \quad m, n \in \mathbb{Z}.$$

We get the triangle $\triangle(3^2n^3 - m, 3^2n^3, 3^2n^3 + m)$, whose area is $(3n)^3$.

In connection with Lemma 2.5, we raise the following question.

Question 4. Is it possible to characterize all the lattice triangles inscribed in the lattice cubic $y = x^3$, whose areas are cubes of natural numbers?

§3. Some open problems

The case of the lattice parabola $y = x^2$ has been treated by Sastry [2], and later, to some extent, by Majumdar [1]. This paper treats the case of the lattice parabola $y = x^3$, in continuation of our previous study in [1]. There are several open problems in connection with the lattice parabola and lattice cubic, some of which are already mentioned in Section 2. It might be a problem of great interest to study the properties of the lattice triangles inscribed in the lattice curve $y = x^n$, where $n \geq 3$ is an integer.

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Eigenvalue problems and inverse problems on SC matrices

Lieya Yan and Minggang Chen
Department of Mathematics, Xi'an University of Architecture and Technology,
Xi'an, 710055, China
E-mail: xajdyly@yahoo.com.cn

Abstract Eigenvalue problem of SC matrices with the form $\alpha\alpha^T$ and matrices sum with the form $\alpha\alpha^T(\alpha \in R^n)$ is discussed, and a structure of orthogonal matrices of SC matrices diagonalized is given. And the method of inverse eigenvalue problem of SC matrices is improved.
Keywords Centrosymmetric matrix, eigenvalue, inverse problem.

Let $J_n = (e_n, e_{n-1}, \cdots, e_1)$, where e_i is the ith column of the $n \times n$ identity matrix. A matrix A is called centrosymmetric if $J_nA = AJ_n$, and anti-centrosymmetric if $J_nA = -AJ_n$. For $A \in R^{n\times n}$, a structure of centrosymmetric matrices and it's solution of the inverse eigenvalue problem were developed by FuZhao Zhou, XiYan Hu and Lei Zhang in [1]. In [3], Trench studied problem of R-symmetric or R-skew symmetric matrices. Recently, some properties and the inverse eigenvalue problem of generalized centrosymmetric matrix have been studied in [4] and [5], respectively.

A centrosymmetric matrix of a real symmetric is called an SC matrix. In this paper, eigenvalue problem of SC matrices with the form $\alpha\alpha^T$ and matrices sum with the form $\alpha\alpha^T(\alpha \in R^n)$ are discussed. Also a structure of orthogonal matrices of SC matrices diagonalized is given. Furthermore the method finding inverse eigenvalue problem of SC matrices given in [2], is improved here.

§1. Eigenvalue problem of SC matrices

Lemma 1.1. Let $0 \neq \alpha \in R^n$. Then $A = \alpha\alpha^T$ has a unique nonzero eigenvalue λ such that $\lambda = \|\alpha\|_2^2$ and a unit eigenvector $x = \dfrac{\alpha}{\|\alpha\|_2}$ corresponding to λ.

Proof. Clearly, $A = \alpha\alpha^T$ is a real symmetric matrix. Since $\alpha \neq 0$, we have rank(A)=1. Therefore, A has a unique nonzero eigenvalue. Denote the eigenvalue of A by λ and the unit eigenvector corresponding to λ by x. Then we get $\alpha\alpha^T x = \lambda x$. Since $\lambda \neq 0$, it follows that

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$\alpha^T x \neq 0$. Let $a = \frac{\lambda}{\alpha^T x}$. Then $\alpha = ax$. Notice that $x^T x = 1$. We have $\lambda = a^2$. Furthermore, $\alpha^T \alpha = \lambda x^T x = a^2$. So $\lambda = a^2 = \alpha^T \alpha = \|\alpha\|_2^2$ and $x = \frac{\alpha}{a} = \frac{\alpha}{\|\alpha\|_2}$.

Corollary 1.2. $A = \alpha\alpha^T$ is a positive semidifinite matrix.

lemma 1.3. Let $0 \neq \alpha \in R^n$. Then $A = \alpha\alpha^T$ is an SC matrix if and only if $J\alpha = \alpha$ or $J\alpha = -\alpha$.

Proof. Because $A = \alpha\alpha^T$ is a real symmetric matrix, we only need to prove that A is a centrosymmetric matrix if and only if $J\alpha = \alpha$ or $J\alpha = -\alpha$.

Suppose that $A = \alpha\alpha^T$ is a centrosymmetric matrix. We have that $x = \frac{\alpha}{\|\alpha\|_2}$ is the unit eigenvector corresponding to the eigenvalue λ of A by Lemma 1.1. Thus, $\alpha = \|\alpha\|_2 x$ and $J\alpha = J(\|\alpha\|_2 x) = \|\alpha\|_2 Jx$. Since $Jx = x$ or $Jx = -x$ from [1], then $J\alpha = \alpha$ or $J\alpha = -\alpha$.

Conversely, suppose $J\alpha = \alpha$ or $J\alpha = -\alpha$, then it is easy to check that $J\alpha\alpha^T J = J\alpha(J\alpha)^T$, i.e., $A = \alpha\alpha^T$ is a centrosymmetric matrix.

Theorem 1.4. Let $0 \neq \alpha_i \in R^n$ and $A_i = \alpha_i \alpha_i^T (i = 1, 2, \cdots, s)$. If $A_i A_j = 0$ for $i, j = 1, 2, \cdots, s$, $i \neq j$, then $A = \sum_{i=1}^{s} A_i$ has only nonzero eigenvalue $\|\alpha_1\|_2^2$, $\|\alpha_2\|_2^2$, $\cdots$, $\|\alpha_s\|_2^2$.

Proof. Clearly, $A_i = \alpha_i \alpha_i^T$ is a real symmetric matrix. It follows that $A = \sum_{i=1}^{s} A_i$ is a real symmetric matrix. Since $\alpha_i \neq 0 (i = 1, 2, \cdots, s)$, we have $\alpha_i \alpha_i^T \neq 0$. Thus $A_i A_j = \alpha_i \alpha_i^T \alpha_j \alpha_j^T = (\alpha_i^T \alpha_j) \alpha_i \alpha_j^T = 0$ if and only if $\alpha_i \alpha_j^T \neq 0$, i.e., α_i and α_j are orthogonal. Consequently, $\alpha_1, \alpha_2, \cdots, \alpha_s$ are orthogonal vectors set.

Meanwhile, $A\alpha_i = A_i \alpha_i = \alpha_i \alpha_i^T \alpha_i = \|\alpha_i\|_2^2 \alpha_i (i = 1, 2, \cdots, s)$, i.e., $\|\alpha_1\|_2^2$, $\|\alpha_2\|_2^2$, $\cdots$, $\|\alpha_s\|_2^2$ are nonzero eigenvalues of A. A is the sum of matrix $A_i (i = 1, 2, \cdots, s)$, where $\mathrm{rank}(A_i) = 1$, and $\mathrm{rank}(A)$ is no more than s. Thus, A has s nonzero eigenvalues at most. Hence, $A = \sum_{i=1}^{s} A_i$ has and has only nonzero eigenvalues $\|\alpha_1\|_2^2$, $\|\alpha_2\|_2^2$, $\cdots$, $\|\alpha_s\|_2^2$.

Corollary 1.5. Let $0 \neq \alpha_i \in R^n$ and $A_i = \alpha_i \alpha_i^T (i = 1, 2, \cdots, s)$. Suppose that $A_i A_j = 0 (i, j = 1, 2, \cdots, s, i \neq j)$. Then $A = \sum_{i=1}^{s} A_i$ is a positive semidifinite matrix where $s < n$, and $A = \sum_{i=1}^{s} A_i$ is a positive difinite matrix where $s = n$.

Corollary 1.6. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be orthonornal column vectors set and $A_i = \alpha_i \alpha_i^T (i = 1, 2, \cdots, s)$. Then $\sum_{i=1}^{s} A_i$ is an identity matrix.

Theorem 1.7. Let A and B be $n \times n$ real symmetric matrices, where $r(A) = r$, $r(B) = s$ ($r(A)$ means rank of A), and $r + s \leq n$, let nonzero eigenvalues of A be $\lambda_1, \lambda_2, \cdots, \lambda_r$, and let nonzero eigenvalues of B be $\mu_1, \mu_2, \cdots, \mu_s$. If $AB = 0$, then nonzero eigenvalues of $A + B$ are $\lambda_1, \lambda_2, \cdots, \lambda_r, \mu_1, \mu_2, \cdots, \mu_s$.

Proof. Since A and B are $n \times n$ real symmetric matrices, then $BA = 0$ where $AB = 0$ and so A and B commute. Hence there exists an orthogonal matrix Q such that $A = Q\Lambda_A Q^T$ and $B = Q\Lambda_B Q^T$, where $\Lambda_A = diag(\lambda_{r_1}, \lambda_{r_2}, \cdots, \lambda_{r_n})$ and $\Lambda_B = diag(\mu_{s_1}, \mu_{s_2}, \cdots, \mu_{s_n})$. Thus, $AB = 0$ if and only if $\Lambda_A \Lambda_B = 0$. $\Lambda_A \Lambda_B = 0$ if and only if $\lambda_{r_i} \mu_{s_i} = 0 (i = 1, 2, \cdots, n)$, that is, if $AB = 0$, then λ_{r_i} and μ_{s_i} equal to zero at least one. Thereby, $\lambda_{r_i} + \mu_{s_i}$ equals

zero or equals λ_j or $\mu_k (j = 1, 2, \cdots, r; k = 1, 2, \cdots, s)$. Meanwhile, since $A + B = Q(\Lambda_A + \Lambda_B)Q^T = Q diag(\lambda_{r_1} + \mu_{s_1}, \lambda_{r_2} + \mu_{s_2}, \cdots, \lambda_{r_n} + \mu_{s_n})Q^T$, if $AB = 0$, then nonzero eigenvalues of $A + B$ are $\lambda_1, \lambda_2, \cdots, \lambda_r, \mu_1, \mu_2, \cdots, \mu_r$. Obviously, $A + B$ is nonsingular where $r + s = n$.

Theorem 1.8. Let $X_1 = (x_1, x_2, \cdots, x_r)$ and $\Lambda_1 = diag(\lambda_1, \lambda_2, \cdots, \lambda_r)$, where $\lambda_1, \lambda_2, \cdots, \lambda_r$ are nonzero and the set $\{x_1, x_2, \cdots, x_r\}$ is an orthonormal vectors set. Then there exists a nonsingular real symmetric matrix A such that $AX_1 = X_1\Lambda_1$.

Proof. Let $A_1 = X_1\Lambda_1 X_1^T$. Since X_1 is a $n \times r$ matrix with orthonormal columns, there exists a $n \times (n - r)$ matrix X_2 with orthonormal column such that $X_2^T X_1 = 0$. Put $A_2 = X_2\Lambda_2 X_2^T$, where $\Lambda_2 = diag(\lambda_{r+1}, \lambda_{r+2}, \cdots, \lambda_n)$ with $det\,\Lambda_2 \neq 0$. Then $A_2A_1 = 0$. Let $A = A_1 + A_2$. By Theorem 1.7, A is nonsingular and

$AX_1 = (A_1 + A_2)X_1 = (X_1\Lambda_1 X_1^T + X_2\Lambda_2 X_2^T)X_1 = X_1\Lambda_1 X_1^T X_1 + X_2\Lambda_2 X_2^T X_1$

$= X_1\Lambda_1 I_2 + 0 = X_1\Lambda_1$.

It is clear that three Theorems above hold for SC matrices.

Theorem 1.9. Let A be SC matrix. Then

(1) For $n = 2k$, $P = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & P_2 \\ -J_kP_1 & J_kP_2 \end{pmatrix}$ is an orthogonal matrix such that

$P^TAP = \begin{pmatrix} \Lambda_1 & \\ & \Lambda_2 \end{pmatrix}$, where P_1, P_2 are $k \times k$ orthogonal matrices and Λ_1, Λ_2 are $k \times k$ diagonal matrices;

(2) For $n = 2k + 1$, $P = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & P_2 \\ 0 & \sqrt{2}\,\gamma^T \\ -J_kP_1 & J_kP_2 \end{pmatrix}$ is an orthogonal matrix such that

$P^TAP = \begin{pmatrix} \Lambda_1 & \\ & \Lambda_2 \end{pmatrix}$, where P_1 is a $k \times k$ orthogonal matrix, $P_2' = \begin{pmatrix} P_2 \\ \gamma^T \end{pmatrix}$ is a $(k+1) \times (k+1)$ orthogonal matrix, $\gamma \in R^{(k+1)\times 1}$, and Λ_1, Λ_2 are $k \times k$ and $(k + 1) \times (k + 1)$ diagonal matrices, respectively.

Proof. If $n = 2k$, then A is a SC matrix and so $A = Q \begin{pmatrix} A_{11} & \\ & A_{22} \end{pmatrix} Q^T$, where

$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & I_k \\ -J_k & J_k \end{pmatrix}$ and A_{11}, A_{22} are $k \times k$ real symmetric matrices. Since A_{11}, A_{22} are $k \times k$ real symmetric matrices, there exist orthogonal matrices P_1, P_2 such that $A_{11} = P_1\Lambda_1 P_1^T$, $A_{22} = P_2\Lambda_2 P_2^T$, where Λ_1,Λ_2 are $k \times k$ diagonal matrices. Let

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & I_k \\ -J_k & J_k \end{pmatrix} \begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & P_2 \\ -J_kP_1 & J_kP_2 \end{pmatrix}.$$

Thus $P^TAP = \begin{pmatrix} \Lambda_1 & \\ & \Lambda_2 \end{pmatrix}$. For the case where $n = 2k + 1$, the proof of the result is similar.

§2. Inverse eigenvalue problems of SC matrices

(1) Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be real numbers. Find an SC matrix A such that $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A.

If $n = 2k$, P_1, P_2 are $k \times k$ orthogonal matrices. Set $P = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & P_2 \\ -J_k P_1 & J_k P_2 \end{pmatrix}$.

Then, $P^T A P = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

If $n = 2k+1$, P_1 is a $k \times k$ orthogonal matrix and $P_2^{'}$ is a $(k+1) \times (k+1)$ orthogonal matrix, where $P_2^{'} = \begin{pmatrix} P_2 \\ \gamma^T \end{pmatrix}$ and $\gamma \in R^{(k+1)\times 1}$. Set

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & P_2 \\ 0 & \sqrt{2}\,\gamma^T \\ -J_k P_1 & J_k\, P_2 \end{pmatrix}.$$

Then $P^T A P = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

Example. 1, 2, 3, 4, 5 are eigenvalues of SC matrix A,

Write $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $P_2 = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{3}} \\ 0 & 1 & 0 \end{pmatrix}$ $\gamma^T = \begin{pmatrix} -\sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}$ $P_2^{'} = \begin{pmatrix} P_2 \\ \gamma^T \end{pmatrix}$.

We can obtain: $A = P diag(1,2,3,4,5) P^T$.

(2) For r given eigenpairs $(\lambda_1, x_1), (\lambda_2, x_2), \cdots, (\lambda_r, x_r)$ of a $n \times n$ SC matrix A, where $x_1, x_2, \cdots, x_r$ is a orthonormal vectors set, find a SC matrix A.

By the characteristic of eigenvector for SC matrix, if $n = 2k$,

let $X = (x_1, x_2, \cdots, x_r) = \frac{1}{\sqrt{2}} \begin{pmatrix} X_{r_1} & X_{r_2} \\ -J_k X_{r_1} & J_k X_{r_2} \end{pmatrix}$, $r_1 + r_2 = r$. Take orthonormal eigenvectors sets $X_{r_1}^{'} \in R^{k\times(k-r_1)}$ and $X_{r_2}^{'} \in R^{k\times(k-r_2)}$, such that $X_{r_1}^T X_{r_1}^{'} = 0$ and $X_{r_2}^T X_{r_2}^{'} = 0$.

Set $P_1 = \begin{pmatrix} X_{r_1} & X_{r_1}^{'} \end{pmatrix}$, $P_2 = \begin{pmatrix} X_{r_2} & X_{r_2}^{'} \end{pmatrix}$.

Then, $P_1 A_{11} P_1^T = diag(\lambda_1, \cdots, \lambda_{r_1}, \lambda_1^{'}, \cdots, \lambda_{k-r_1}^{'})$,

$P_2 A_{22} P_2^T = diag(\lambda_{r_1+1}, \cdots, \lambda_r, \lambda_{r_1+1}^{'}, \cdots, \lambda_{k-r_2}^{'})$, where$\lambda_1^{'}, \cdots, \lambda_{k-r_1}^{'}$ $\lambda_{r_1+1}^{'}, \cdots, \lambda_{k-r_2}^{'}$ are arbitrary real numbers.

By Theorem 1.9, if $P = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & P_2 \\ -J_k P_1 & J_k P_2 \end{pmatrix}$, then

$A = P diag(\lambda_1, \cdots, \lambda_{r_1}, \lambda_1^{'}, \cdots, \lambda_{k-r_1}^{'}, \lambda_{r_1+1}, \cdots, \lambda_r, \lambda_{r_1+1}^{'}, \cdots, \lambda_{k-r_2}^{'}) P^T$;

If $n = 2k+1$, without loss of generality , we suppose that

$$X = (x_1, x_2, \cdots, x_r) = \frac{1}{\sqrt{2}} \begin{pmatrix} X_{r_1} & X_{r_2} \\ 0 & \sqrt{2}\,\gamma^T \\ -J_k X_{r_1} & J_k X_{r_2} \end{pmatrix}, r_1 + r_2 = r.$$

The result will be obtained.

Example. For two given eigenvalues 1, 3 of 4×4 SC matrix A, the eigenvectors corresponding to the eigenvalues are $X_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1, & 0, & 0, & -1 \end{pmatrix}^T$ and $X_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0, & 1, & 1, & 0 \end{pmatrix}^T$. Take $X_1^{'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0, & 1, & -1, & 0 \end{pmatrix}^T$ and $X_2^{'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1, & 0, & 0, & 1 \end{pmatrix}^T$.
Set $P = (X_1, X_1^{'}, X_2, X_2^{'})$. Then $A = Pdiag(1, \lambda_1^{'}, 3, \lambda_2^{'})P^T$, $\lambda_1^{'}$, $\lambda_2^{'}$ are arbitrary real numbers.

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On some Smarandache determinant sequences

A.A.K. Majumdar

Beppu-shi 875-8577, Oita-ken, Japan
majumdar@apu.ac.jp aakmajumdar@gmail.com

Abstract Murthy [1] introduced the concept of the Smarandache Cyclic Determinant Natural Sequence, the Smarandache Cyclic Arithmetic Determinant Sequence, the Smarandache Bisymmetric Determinant Natural Sequence, and the Smarandache Bisymmetric Arithmetic Determinant Sequence. In this paper, we derive the n-th terms of these four sequences.

Keywords The Smarandache cyclic determinant natural sequence, the Smarandache cyclic arithmetic determinant sequence, the Smarandache bisymmetric determinant natural sequence, the Smarandache bisymmetric arithmetic determinant sequence.

§1. Introduction

Murthy [1] introduced the concept of the Smarandache cyclic determinant natural sequence, the Smarandache cyclic arithmetic determinant sequence, the Smarandache bisymmetric determinant natural sequence, and the Smarandache bisymmetric arithmetic determinant sequence as follows.

Definition 1.1. The Smarandache cyclic determinant natural sequence, $\{SCDNS(n)\}$ is

$$\left\{ |1|, \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}, \ ... \right\}.$$

Murthy conjectured that the $n-th$ term of the above sequence is

$$SCDNS(n) = (-1)^{\left[\frac{n}{2}\right]} \ \frac{\mathrm{n}+1}{2} \ n^{n-1},$$

where $[x]$ denotes the greatest integer less than or equal to x .

Definition 1.2. The Smarandache cyclic arithmetic determinant sequence, $\{SCADS(n)\}$ is

$$\left\{ |1|, \begin{vmatrix} a & a+d \\ a+d & a \end{vmatrix}, \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a \\ a+2d & a & a+d \end{vmatrix}, \ ... \right\}.$$

Murthy conjectured, erroneously, that the $n-th$ term of the above sequence is

$$SCDNS(n) = (-1)^{\left[\frac{n}{2}\right]} \ \frac{\mathrm{a} + (\mathrm{n}-1)d}{2} \ (nd)^{n-1}$$

where $[x]$ denotes the greatest integer less than or equal to x .

Definition 1.3. The Smarandache bisymmetric determinant natural sequence, $\{SBDNS(n)\}$ is

$$\left\{ |1|, \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 3 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix}, \ ... \right\}.$$

Definition 1.4. The Smarandache bisymmetric arithmetic determinant sequence, $\{SBADS(n)\}$ is

$$\left\{ |1|, \begin{vmatrix} a & a+d \\ a+d & a \end{vmatrix}, \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a+d \\ a+2d & a+d & a \end{vmatrix}, \ ... \right\}.$$

Murthy also conjectured about the n-th terms of the last two sequences, but those expressions are not correct.

In this paper, we derive explicit forms of the n-th terms of the four sequences. These are given in Section 3. Some preliminary results, that would be necessary in the derivation of the expressions of the n-th terms of the sequences, are given in Section 2.

§2. Some preliminary results

In this section, we derive some results that would be needed later in proving the main results of this paper in Section 3. We start with the following result.

Lemma 2.1. Let $D \equiv |d_{ij}|$ be the determinant of order $n \geq 2$ with

$$d_{ij} = \begin{cases} a, \text{ if i = j} \geqslant 2; \\ 1, \text{ otherwise.} \end{cases}$$

where a is a fixed number. Then,

$$D \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix} = (a-1)^{n-1}.$$

Proof. Performing the indicated column operations (where $C_i \to C_i - C_1$ indicates the column operation of subtracting the 1st column from the ith column, $2 \leq i \leq n$), we get

$$D \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix} \begin{matrix} = \\ C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \\ \vdots \\ C_n \to C_n - C_1 \\ \end{matrix} \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & a-1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & a-1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 1 & 0 & 0 & \cdots & a-1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & a-1 \end{vmatrix}$$

$$= \begin{vmatrix} a-1 & 0 & \cdots & 0 & 0 \\ 0 & a-1 & \cdots & 0 & 0 \\ & & & & \\ 0 & 0 & \cdots & a-1 & 0 \\ 0 & 0 & \cdots & 0 & a-1 \end{vmatrix},$$

which is a determinant of order $n-1$ whose diagonal elements are all $a-1$ and off-diagonal elements are all zero. Hence,

$$D = (a-1)^{n-1}.$$

Lemma 2.2. Let $D^a = \left|d_{ij}^{(a)}\right|$ be the determinant of order $n \geq 2$ whose diagonal elements are all a (where a is a fixed number) and off-diagonal elements are all 1, that is,

$$d_{ij} = \begin{cases} a, \text{ if i = j} \geqslant 1; \\ 1, \text{ otherwise.} \end{cases}$$

Then,

$$D^{(a)} \equiv \begin{vmatrix} a & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix} = (a-1)^{n-1}(a+n-1).$$

Proof. We perform the indicated column operations (where $C_1 \to C_1 + C_2 + ... + C_n$ indicates the operation of adding all the columns and then replacing the 1st column by that sum, and $C_1 \to \dfrac{1}{C_1 + C_2 + ... + C_n}$ denotes the operation of taking out the common sum) to get

$$D^{(a)} \equiv \begin{vmatrix} a & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \end{matrix} \begin{matrix} = \\ C_1 \to C_1 + C_2 + ... + C_n \\ C_1 \to \dfrac{1}{C_1 + C_2 + ... + C_n} \end{matrix} (a+n-1) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & a \end{vmatrix}$$

$$= (a+n-1)(a-1)^{n-1},$$

where the last equality is by virtue of Lemma 2.1.

Corollary 2.1. The value of the following determinant of order $n \geq 2$ is

$$\begin{vmatrix} -n & 1 & 1 & \cdots & 1 & 1 \\ 1 & -n & 1 & \cdots & 1 & 1 \\ 1 & 1 & -n & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & -n & 1 \\ 1 & 1 & 1 & \cdots & 1 & -n \end{vmatrix} = (-1)^n \, (n+1)^{n-1}.$$

Proof. Follows immediately from Lemma 2.2 as a particular case when $a = -n$.

Lemma 2.3. Let $A_n = |a_{ij}|$ be the determinant of order $n \geq 2$, defined by

$$a_{ij} = \begin{cases} 1, \text{ if i} \leqslant \text{j}; \\ -1, \text{ otherwise.} \end{cases}$$

Then,

$$A_n \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & & \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix} = 2^{n-1}.$$

Proof. The proof is by induction on n. Since

$$A_2 = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2,$$

the result is true for $n = 2$. So, we assume the validity of the result for some integer $n \geq 2$. To prove the result for $n+1$, we consider the determinant of order $n+1$, and perform the indicated

column operations (where $C_1 \to C_1 + C_n$ indicates the operation of adding the $n-th$ column to the 1^{st} column to get the new 1^{st} column), to get

$$A_{n+1} \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & & \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix} \begin{matrix} \\ = \\ C_1 \to C_1 + C_n \\ \\ \\ \\ \end{matrix} \begin{vmatrix} 2 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & & \\ 0 & -1 & -1 & \cdots & -1 & 1 & 1 \\ 0 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & & \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix} = 2A_n = 2^n,$$

by virtue of the induction hypothesis. Thus, the result is true for $n+1$, which completes induction.

Corollary 2.2. The value of the following determinant of order $n \geq 2$ is

$$B_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & -1 & -1 \\ \vdots & & & & & & \\ 1 & 1 & -1 & \cdots & -1 & -1 & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 & -1 \end{vmatrix} = (-1)^{\left\lceil \frac{n}{2} \right\rceil} \, 2^{n-1}.$$

Proof. To prove the result, note that the determinant B_n can be obtained from the determinant A_n of Lemma 2.3 by successive interchange of columns. To get the determinant B_n from the determinant A_n, we consider the two cases depending on whether n is even or odd.

Case 1 : When n is even, say, $n = 2m$ for some integer $m \geq 1$.

In this case, starting with the determinant $B_n = B_{2m}$, we perform the indicated column operations.

$$B_n = B_{2m} = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & -1 \\ \vdots & & & & & \\ 1 & 1 & \cdots & -1 & -1 & -1 \\ 1 & -1 & \cdots & -1 & -1 & -1 \end{vmatrix} \begin{matrix} \\ = \\ C_1 \to C_{2m} \\ C_2 \to C_{2m-1} \\ \vdots \\ C_m \to C_{m+1} \end{matrix} (-1)^m \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & \\ -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix}$$

$$= (-1)^m 2^{n-1}.$$

Case 2 : When n is odd, say, $n = 2m + 1$ for some integer $m \geq 1$. In this case,

$$B_n = B_{2m+1} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & -1 & -1 \\ \vdots & & & & & & \\ 1 & 1 & -1 & \cdots & -1 & -1 & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 & -1 \end{vmatrix}$$

$$\begin{matrix} = \\ C_1 \to C_{2m+1} \\ C_2 \to C_{2m} \\ \vdots \\ C_m \to C_{m+2} \end{matrix} \quad (-1)^m \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & -1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & & & & & & \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{vmatrix} = (-1)^m 2^{n-1}.$$

Since, in either case, $m = \left\lceil \frac{n}{2} \right\rceil$, the result is established.

§3. Main results

In this section, we derive the explicit expressions of the n-th terms of the four determinant sequences, namely, the Smarandache cyclic determinant natural sequence, the Smarandache cyclic arithmetic determinant sequence, the Smarandache bisymmetric determinant natural sequence, and the Smarandache bisymmetric arithmetic determinant sequence. These are given in Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4 respectively.

Theorem 3.1. The n-th term of the Smarandache cyclic determinant natural sequence, $SCDNS(n)$ is

$$SCDNS(n) = \begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & 5 & \cdots & n-1 & n & 1 \\ 3 & 4 & 5 & 6 & \cdots & n & 1 & 2 \\ 4 & 5 & 6 & 7 & \cdots & 1 & 2 & 3 \\ \vdots & & & & & & & \\ n-1 & n & 1 & 2 & \cdots & n-4 & n-3 & n-2 \\ n & 1 & 2 & 3 & \cdots & n-3 & n-2 & n-1 \end{vmatrix} = (-1)^{\left\lceil \frac{n}{2} \right\rceil} \frac{n+1}{2} n^{n-1}.$$

Proof. We consider separately the possible two cases.

Case 1 : When n is even, say, $n = 2m$ for some integer $m \geq 1$ (so that $\left\lceil \frac{n}{2} \right\rceil = m$).
We now perform the indicated operations on $SCDNS(n)$ (where $C_i \leftrightarrow C_j$ denotes the operation

of interchanging the i -thcolumn and the j-th column, and $R_i \to R_i - R_j$ means that the j-th row is subtracted from the i-th row to get the new i-th row). Note that, there are in total, m interchanges of columns, each changing the value of $SCDNS(n)$ by -1. Then,

$$SCDNS(n) = \begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & 2m-2 & 2m-1 & 2m \\ 2 & 3 & 4 & 5 & \cdots & 2m-1 & 2m & 1 \\ 3 & 4 & 5 & 6 & \cdots & 2m & 1 & 2 \\ 4 & 5 & 6 & 7 & \cdots & 1 & 2 & 3 \\ \vdots & & & & & & & \\ 2m-1 & 2m & 1 & 2 & \cdots & 2m-4 & 2m-3 & 2m-2 \\ 2m & 1 & 2 & 3 & \cdots & 2m-3 & 2m-2 & 2m-1 \end{vmatrix}$$

$$\begin{matrix} = \\ C_1 \leftrightarrow C_{2m} \\ C_2 \leftrightarrow C_{2m-1} \\ \vdots \\ C_m \leftrightarrow C_{m+1} \end{matrix} \quad (-1)^m \begin{vmatrix} 2m & 2m-1 & 2m-2 & \cdots & 3 & 2 & 1 \\ 1 & 2m & 2m-1 & \cdots & 4 & 3 & 2 \\ 2 & 1 & 2m & \cdots & 5 & 4 & 3 \\ 3 & 2 & 1 & \cdots & 6 & 5 & 4 \\ \vdots & & & & & & \\ 2m-3 & 2m-4 & 2m-5 & & 2m & 2m-1 & 2m-2 \\ 2m-2 & 2m-3 & 2m-4 & \cdots & 1 & 2m & 2m-1 \\ 2m-1 & 2m-2 & 2m-3 & \cdots & 2 & 1 & 2m \end{vmatrix}$$

$$\begin{matrix} = \\ C_{2m} \to C_1 + C_2 + ... + C_{2m} \\ C_{2m} \to \dfrac{1}{C_1 + C_2 + ... + C_{2m}} \end{matrix} \quad (-1)^m \, \frac{2m(2m+1)}{2} \begin{vmatrix} 2m & 2m-1 & \cdots & 3 & 2 & 1 \\ 1 & 2m & \cdots & 4 & 3 & 1 \\ \vdots & & & & & \\ 2m-3 & 2m-4 & & 2m & 2m-1 & 1 \\ 2m-2 & 2m-3 & \cdots & 1 & 2m & 1 \\ 2m-1 & 2m-2 & \cdots & 2 & 1 & 1 \end{vmatrix}$$

$$\begin{matrix} = \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \\ \vdots \\ R_{2m} \to R_{2m} - R_{2m-1} \end{matrix} \quad (-1)^m \frac{2m(2m+1)}{2} \begin{vmatrix} 2m & 2m-1 & 2m-2 & \cdots & 2 & 1 \\ 1-2m & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1-2m & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1-2m & \cdots & 1 & 0 \\ \vdots & & & & & \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1-2m & 0 \end{vmatrix}$$

$$= (-1)^m \frac{2m(2m+1)}{2} (-1)^{2m+1} \begin{vmatrix} 1-2m & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1-2m & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1-2m & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1-2m & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1-2m \end{vmatrix}$$

$$= (-1)^{m+1} \frac{2m(2m+1)}{2} \left\{ (-1)^{2m-1} (2m)^{2(m-1)} \right\} = (-1)^m \frac{2m+1}{2} (2m)^{2m-1}.$$

Case 2 : When n is odd, say, $n = 2m+1$ for some integer $m \geq 1$ (so that $\left[\frac{n}{2}\right] = m$). Here,

$$SCDNS(n) = \begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & 2m-1 & 2m & 2m+1 \\ 2 & 3 & 4 & 5 & \cdots & 2m & 2m+1 & 1 \\ 3 & 4 & 5 & 6 & \cdots & 2m+1 & 1 & 2 \\ 4 & 5 & 6 & 7 & \cdots & 1 & 2 & 3 \\ \vdots & & & & & & & \\ 2m & 2m+1 & 1 & 2 & \cdots & 2m-3 & 2m-2 & 2m-1 \\ 2m+1 & 1 & 2 & 3 & \cdots & 2m-2 & 2m-1 & 2m \end{vmatrix}$$

$$= (-1)^m \begin{vmatrix} 2m+1 & 2m & 2m-1 & \cdots & 3 & 2 & 1 \\ 1 & 2m+1 & 2m & \cdots & 4 & 3 & 2 \\ 2 & 1 & 2m+1 & \cdots & 5 & 4 & 3 \\ 3 & 2 & 1 & \cdots & 6 & 5 & 4 \\ \vdots & & & & & & \\ 2m-2 & 2m-3 & 2m-4 & & 2m+1 & 2m & 2m-1 \\ 2m-1 & 2m-2 & 2m-3 & \cdots & 1 & 2m+1 & 2m \\ 2m & 2m-1 & 2m-2 & \cdots & 2 & 1 & 2m+1 \end{vmatrix}$$

$C_1 \leftrightarrow C_{2m+1}$

$C_2 \leftrightarrow C_{2m}$

$C_3 \leftrightarrow C_{2m-1}$

$\vdots$

$C_m \leftrightarrow C_{m+2}$

$$=$$
$$C_{2m+1} \to C_1 + C_2 + ... + C_{2m+1}$$
$$C_{2m+1} \to \frac{1}{C_1 + C_2 + ... + C_{2m+1}}$$

$$(-1)^m \frac{(2m+1)(2m+2)}{2} \begin{vmatrix} 2m+1 & 2m & 2m-1 & \cdots & 3 & 2 & 1 \\ 1 & 2m+1 & 2m & \cdots & 4 & 3 & 1 \\ 2 & 1 & 2m+1 & \cdots & 5 & 4 & 1 \\ 3 & 2 & 1 & \cdots & 6 & 5 & 1 \\ \vdots & & & & & & \\ 2m-2 & 2m-3 & 2m-4 & & 2m+1 & 2m & 1 \\ 2m-1 & 2m-2 & 2m-3 & \cdots & 1 & 2m+1 & 1 \\ 2m & 2m-1 & 2m-2 & \cdots & 2 & 1 & 1 \end{vmatrix}$$

$$= \quad (-1)^m \frac{2m(2m+1)}{2} \begin{vmatrix} 2m+1 & 2m & 2m-1 & \cdots & 3 & 2 & 1 \\ -2m & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & -2m & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & -2m & \cdots & 1 & 1 & 0 \\ \vdots & & & & & & \\ 1 & 1 & 1 & & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & -2m & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & -2m & 0 \end{vmatrix}$$

$$R_2 \to R_2 - R_1$$
$$R_3 \to R_3 - R_2$$
$$\vdots$$
$$R_{2m+1} \to R_{2m+1} - R_{2m}$$

$$= (-1)^m \frac{(2m+1)(2m+2)}{2} (-1)^{2m+2} \begin{vmatrix} -2m & 1 & 1 & \cdots & 1 & 1 \\ 1 & -2m & 1 & \cdots & 1 & 1 \\ 1 & 1 & -2m & \cdots & 1 & 1 \\ \vdots & & & & & \\ 1 & 1 & 1 & & 1 & 1 \\ 1 & 1 & 1 & \cdots & -2m & 1 \\ 1 & 1 & 1 & \cdots & 1 & -2m \end{vmatrix}$$

$$= (-1)^m \frac{(2m+1)(2m+2)}{2} \left\{(-1)^{2m}(2m+1)^{2m-1}\right\} = (-1)^m \frac{2m+2}{2}(2m+1)^{2m}.$$

Thus, the result is true both when n is even and when n is odd, completing the proof.

Theorem 3.2. The n-th term of the Smarandache cyclic arithmetic determinant sequence,

$SCADS(n)$ is

$$SCADS(n) = \begin{vmatrix} a & a+d & a+2d & \cdots & a+(n-2)d & a+(n-1)d \\ a+d & a+2d & a+3d & \cdots & a+(n-1)d & a \\ a+2d & a+3d & a+4d & \cdots & a & a+d \\ a+3d & a+4d & a+5d & \cdots & a+d & a+2d \\ \vdots & & & & & \\ a+(n-2)d & a+(n-1)d & a & \cdots & a+(n-4)d & a+(n-3)d \\ a+(n-1)d & a & a+d & \cdots & a+(n-3)d & a+(n-2)d \end{vmatrix}$$

$$= (-1)^{\left\lceil \frac{n}{2} \right\rceil} \left(a + \frac{n-1}{2}d\right) (nd)^{n-1}.$$

Proof. Here also, we consider separately the possible two cases.

Case 1 : If $n = 2m$ for some integer $m \geq 1$ (so that $\left\lceil \frac{n}{2} \right\rceil = m$). In this case, performing the indicated column and row operations, we get successively

$$SCADS(n) = \begin{vmatrix} a & a+d & \cdots & a+(2m-2)d & a+(2m-1)d \\ a+d & a+2d & \cdots & a+(2m-1)d & a \\ a+2d & a+3d & \cdots & a & a+d \\ a+3d & a+4d & \cdots & a+d & a+2d \\ \vdots & & & & \\ a+(2m-2)d & a+(2m-1)d & \cdots & a+(2m-4)d & a+(2m-3)d \\ a+(2m-1)d & a & \cdots & a+(2m-3)d & a+(2m-2)d \end{vmatrix}$$

$$\begin{matrix} = \\ C_1 \leftrightarrow C_{2m} \\ C_2 \leftrightarrow C_{2m-1} \\ \vdots \\ C_m \leftrightarrow C_{m+1} \end{matrix} \quad (-1)^m \begin{vmatrix} a+(2m-1)d & a+(2m-2)d & \cdots & a+d & a \\ a & a+(2m-1)d & \cdots & a+2d & a+d \\ a+d & a & \cdots & a+3d & a+2d \\ a+2d & a+d & \cdots & a+4d & a+3d \\ \vdots & & & & \\ a+(2m-4)d & a+(2m-5)d & & a+(2m-2)d & a+(2m-3)d \\ a+(2m-3)d & a+(2m-4)d & \cdots & a+(2m-1)d & a+(2m-2)d \\ a+(2m-2)d & a+(2m-3)d & \cdots & a & a+(2m-1)d \end{vmatrix}$$

$$=$$

$$C_{2m} \to C_1 + C_2 + ... + C_{2m}$$

$$C_{2m} \to \frac{1}{C_1 + C_2 + ... + C_{2m}}$$

$$(-1)^m S_{2m} \begin{vmatrix} a+(2m-1)d & a+(2m-2)d & \cdots & a+d & 1 \\ a & a+(2m-1)d & \cdots & a+2d & 1 \\ a+d & a & \cdots & a+3d & 1 \\ a+2d & a+d & \cdots & a+4d & 1 \\ \vdots & & & & \\ a+(2m-4)d & a+(2m-5)d & & a+(2m-2)d & 1 \\ a+(2m-3)d & a+(2m-4)d & \cdots & a+(2m-1)d & 1 \\ a+(2m-2)d & a+(2m-3)d & \cdots & a & 1 \end{vmatrix}$$

$$\left(S_{2m} = a + (a+d) + (a+2d) + \ldots + \{a+(2m-1)d\} = 2ma + \frac{2m(2m-1)}{2}d\right)$$

$$\begin{array}{rl} = & (-1)^m \; S_{2\mathrm{m}} \\ R_2 \to R_2 - R_1 & \\ R_3 \to R_3 - R_2 & \\ \vdots & \\ R_{2m} \to R_{2m} - R_{2m-1} & \end{array} \begin{vmatrix} a+(2m-1)d & a+(2m-2)d & \cdots & a+d & 1 \\ (1-2m)d & d & \cdots & d & 0 \\ d & (1-2m)d & \cdots & d & 0 \\ d & d & \cdots & d & 0 \\ \vdots & & & & \\ d & d & \cdots & d & 0 \\ d & d & \cdots & (1-2m)d & 0 \end{vmatrix}$$

$$= (-1)^m \left\{2ma + \frac{2m(2m-1)}{2}d\right\} \; (-1)^{2m+1} \; d^{2m-1} \begin{vmatrix} 1-2m & 1 & \cdots & 1 \\ 1 & 1-2m & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1-2m \end{vmatrix}$$

$$\begin{aligned} &= (-1)^{m+1} \left\{2ma + \frac{2m(2m-1)}{2}d\right\} \left\{d^{2m-1}(-1)^{2m-1}(2m)^{2(m-1)}\right\} \\ &= (-1)^m \left\{a + (\frac{2m-1}{2})d\right\} \; d^{2m-1} \; (2m)^{2m-1}. \end{aligned}$$

Case 2 : If $n = 2m+1$ for some integer $m \geq 1$ (so that $\left[\frac{n}{2}\right] = m$).

In this case, performing the indicated column and row operations, we get successively

$$\text{SCADS(n)} = \begin{vmatrix} a & a+d & a+2d & \cdots & a+(2m-1)d & a+2md \\ a+d & a+2d & a+3d & \cdots & a+2md & a \\ a+2d & a+3d & a+4d & \cdots & a & a+d \\ a+3d & a+4d & a+5d & \cdots & a+d & a+2d \\ \vdots & & & & & \\ a+(2m-1)d & a+2md & a & \cdots & a+(2m-3)d & a+(2m-2)d \\ a+2md & a & a+d & \cdots & a+(2m-2)d & a+(2m-1)d \end{vmatrix}$$

$$=$$

$$C_1 \leftrightarrow C_{2m+1}$$
$$C_2 \leftrightarrow C_{2m}$$
$$\vdots$$
$$C_m \leftrightarrow C_{m+2}$$

$$(-1)^m \begin{vmatrix} a+2md & a+(2m-1)d & \cdots & a+d & a \\ a & a+2md & \cdots & a+2d & a+d \\ a+d & a & \cdots & a+3d & a+2d \\ a+2d & a+d & \cdots & a+4d & a+3d \\ \vdots & & & & \\ a+(2m-3)d & a+(2m-4)d & & a+(2m-1)d & a+(2m-2)d \\ a+(2m-2)d & a+(2m-3)d & \cdots & a+2md & a+(2m-1)d \\ a+(2m-1)d & a+(2m-2)d & \cdots & a & a+2md \end{vmatrix}$$

$$=$$

$$C_{2m+1} \to C_1 + C_2 ... + C_{2m+1}$$
$$C_{2m+1} \to \frac{1}{C_1 + C_2 ... + C_{2m+1}}$$

$$(-1)^m \, S_{2m+1} \begin{vmatrix} a+2md & a+(2m-1)d & \cdots & a+2d & a+d & 1 \\ a & a+2md & \cdots & a+3d & a+2d & 1 \\ a+d & a & \cdots & a+4d & a+3d & 1 \\ a+2d & a+d & \cdots & a+5d & a+4d & 1 \\ \vdots & & & & & \\ a+(2m-2)d & a+(2m-3)d & & a & a+2md & 1 \\ a+(2m-1)d & a+(2m-2)d & \cdots & a+d & a & 1 \end{vmatrix}$$

$$\left(S_{2m+1} = a + (a+d) + (a+2d) \ldots + (a+2md) = (2m+1)a + \frac{2m(2m+1)}{2} d \right)$$

$$
\begin{array}{c}
= \\
R_2 \to R_2 - R_1 \\
R_3 \to R_3 - R_2 \\
\vdots \\
R_{2m+1} \to R_{2m+1} - R_{2m}
\end{array}
\quad (-1)^m \, S_{2m+1}
\begin{vmatrix}
a+2md & a+(2m-1)d & \cdots & a+d & 1 \\
-2md & d & \cdots & d & 0 \\
d & -2md & \cdots & d & 0 \\
\vdots & & & & \\
d & d & & d & 0 \\
d & d & \cdots & -2md & 0
\end{vmatrix}
$$

$$
= (-1)^m \left\{ (2m+1)a + \frac{2m(2m+1)}{2} d \right\} (-1)^{2m+2} d^{2m}
\begin{vmatrix}
-2m & 1 & \cdots & 1 \\
1 & -2m & \cdots & 1 \\
\vdots & & & \\
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & -2m
\end{vmatrix}
$$

$$
\begin{aligned}
&= (-1)^m \left\{ (2m+1)a + \frac{2m(2m+1)}{2} d \right\} \left\{ d^{2m} (-1)^{2m} (2m+1)^{2m-1} \right\} \\
&= (-1)^m \left\{ a + \frac{2m}{2} d \right\} d^{2m} (2m+1)^{2m}.
\end{aligned}
$$

Thus, in both the cases, the result holds true. This completes the proof.

Theorem 3.3. The n-th term of the Smarandache bisymmetric determinant natural sequence, $\{SBDNS(n)\}$, $n \geq 5$, is

$$
SBDNS(n) =
\begin{vmatrix}
1 & 2 & 3 & \cdots & n-1 & n \\
2 & 3 & 4 & \cdots & n & n-1 \\
3 & 4 & 5 & \cdots & n-1 & n-2 \\
\vdots & & & & & \\
n-2 & n-1 & n & & 4 & 3 \\
n-1 & n & n-1 & \cdots & 3 & 2 \\
n & n-1 & n-2 & \cdots & 2 & 1
\end{vmatrix}
= (-1)^{\left\lceil \frac{n}{2} \right\rceil} (n+1) 2^{n-2}.
$$

Proof. We perform the indicated row and column operations to reduce the determinant

$SBDNS(n)$ to the form B_{n-1} (of Corollary 2.2) as follows :

$$SBDNS(n) = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & \cdots & n-1 & n & n-1 \\ 3 & 4 & 5 & \cdots & n & n-1 & n-2 \\ \vdots & & & & & & \\ n-2 & n-1 & n & & 5 & 4 & 3 \\ n-1 & n & n-1 & \cdots & 4 & 3 & 2 \\ n & n-1 & n-2 & \cdots & 3 & 2 & 1 \end{vmatrix}$$

$$\begin{matrix} \\ = \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \\ \vdots \\ R_n \to R_n - R_{n-1} \end{matrix} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 1 & 1 & 1 & \cdots & 1 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 & -1 & -1 \\ \vdots & & & & & & \\ 1 & 1 & 1 & & -1 & -1 & -1 \\ 1 & 1 & -1 & \cdots & -1 & - & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 & -1 \end{vmatrix}$$

$$\begin{matrix} \\ = \\ C_n \to C_n + C_1 \end{matrix} \begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n+1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & -1 & 0 \\ \vdots & & & & & & \\ 1 & 1 & 1 & & -1 & -1 & 0 \\ 1 & 1 & -1 & \cdots & -1 & -1 & 0 \\ 1 & -1 & -1 & \cdots & -1 & -1 & 0 \end{vmatrix}$$

$$= (-1)^{n+1}(n+1) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & -1 \\ \vdots & & & & & \\ 1 & 1 & 1 & & -1 & -1 \\ 1 & 1 & -1 & \cdots & -1 & -1 \\ 1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix} = (-1)^{n+1}(n+1)B_{n-1}$$

$$= (-1)^{n+1}(n+1) \left\{ (-1)^{\left[\frac{n-1}{2}\right]} 2^{n-2} \right\}.$$

Now, if $n = 2m+1$, then $(-1)^{n+1+\left[\frac{n-1}{2}\right]} = (-1)^{(2m+2)+m} = (-1)^m = (-1)^{\left[\frac{n}{2}\right]}$, and if $n = 2m$, then $(-1)^{n+1+\left[\frac{n-1}{2}\right]} = (-1)^{2m+1+(m-1)} = (-1)^m = (-1)^{\left[\frac{n}{2}\right]}$.

Hence, finally, we get $SBDNS(n) = (-1)^{\left\lceil \frac{n}{2} \right\rceil}(n+1)2^{n-2}$.

Remark 3.1. The values of $SBDNS(3)$ and $SBDNS(4)$ can be obtained by proceeding as in Theorem 3.3. Thus,

$$SBDNS(3) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \end{matrix} = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} \begin{matrix} \\ C_3 \to C_3 + C_1 \end{matrix} = \begin{vmatrix} 1 & 2 & 4 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -8,$$

$$SBDNS(4) = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 3 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_3 \end{matrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{vmatrix}$$

$$\begin{matrix} = \\ C_4 \to C_4 + C_1 \end{matrix} \begin{vmatrix} 1 & 2 & 3 & 5 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{vmatrix} = (-5) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} = (-5)\{(-1)^{\left[\frac{3}{2}\right]}2^{3-1}\} = 20.$$

Theorem 3.4. The n-th term of the Smarandache bisymmetric arithmetic determinant sequence, $\{SBADS(n)\}$, $n \geq 5$, is

$$\text{SBADS(n)} = \begin{vmatrix} a & a+d & \cdots & a+(n-2)d & a(n-1)d \\ a+d & a+2d & \cdots & a+(n-1)d & a+(n-2)d \\ \vdots & & & & \\ a+(n-2)d & a+(n-1)d & \cdots & a+2d & a+d \\ a+(n-1)d & a+(n-2)d & \cdots & a+d & a \end{vmatrix}$$

$$= (-1)^{\left\lceil \frac{n}{2} \right\rceil} \left(a + \frac{n-1}{2}d\right)(2d)^{n-1}.$$

Proof. We get the desired result, starting from $SBADS(n)$, expressing this in terms of the determinant B_{n-1} (of Corollary 2.2) by performing the indicated row and column operations.

$$\text{SBADS(n)} = \begin{vmatrix} a & a+d & \cdots & a+(n-2)d & a+(n-1)d \\ a+d & a+2d & \cdots & a+(n-1)d & a+(n-2)d \\ \vdots & & & & \\ a(n-2)d & a+(n-1)d & \cdots & a+2d & a+d \\ a+(n-1)d & a+(n-2)d & \cdots & a+d & a \end{vmatrix}$$

$$
\begin{array}{c}
= \\
R_2 \to R_2 - R_1 \\
R_3 \to R_3 - R_2 \\
\vdots \\
R_n \to R_n - R_{n-1}
\end{array}
\begin{vmatrix}
a & a+d & a+2d & \cdots & a+(n-3)d & a+(n-2)d & a+(n-1)d \\
\mathrm{d} & d & d & \cdots & d & d & -d \\
\mathrm{d} & d & d & \cdots & d & -d & -d \\
\vdots & & & & & & \\
d & d & -d & \cdots & -d & -d & -d \\
d & -d & -d & \cdots & -d & -d & -d
\end{vmatrix}
$$

$$
\begin{array}{c}
= \quad d^{n-1} \\
C_n \to C_n + C_1
\end{array}
\begin{vmatrix}
a & a+d & a+2d & \cdots & a+(n-3)d & a+(n-2)d & 2a+(n-1)d \\
1 & 1 & 1 & \cdots 1 & 1 & 0 & \\
1 & 1 & 1 & \cdots & 1 & -1 & 0 \\
\vdots & & & & & & \\
1 & 1 & 1 & & -1 & -1 & 0 \\
1 & 1 & -1 & \cdots & -1 & -1 & 0 \\
1 & -1 & -1 & \cdots & -1 & -1 & 0
\end{vmatrix}
$$

$$
= (-1)^{n+1}\, d^{n-1}\, \{2a+(n-1)d\}
\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & \cdots & 1 & -1 \\
\vdots & & & & & \\
1 & 1 & 1 & & -1 & -1 \\
1 & 1 & -1 & \cdots & -1 & -1 \\
1 & -1 & -1 & \cdots & -1 & -1
\end{vmatrix}
$$

$$
= (-1)^{n+1} d^{n-1}\, \{2a+(n-1)d\} \left((-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} 2^{n-2} \right) = (-1)^{\left\lceil \frac{n}{2} \right\rceil} d^{n-1}\, \{2a+(n-1)d\}\, 2^{n-2}.
$$

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Congruences on Clifford quasi-regular semigroups[1]

Xueming Ren † and Xu Dong Wang‡
Department of Mathematics, Xi'an University of Architecture
and Technology Xi'an 710055, P. R. China
E-mail: xmren@xauat.edu.cn
E-mail: wangxudong401@sina.com

Abstract Let S be a nil-extension of a Clifford semigroup K by a nil semigroup Q. A congruence pair (δ, ω) on S consists of a congruence δ on Q and a congruence ω on K. It is proved that there is an order-preserving bijection $\Gamma : \sigma \mapsto (\sigma_Q, \sigma_K)$ from the set of all congruences on S onto the set of all congruence pairs on S, where σ_K is the restriction of σ on K, $\sigma_Q = (\sigma \vee \rho_K)/\rho_K$ and ρ_K is the Rees congruence on S induced by K.

Keywords Clifford quasi-regular semigroups, Nil-extension, congruence pairs.

§1. Introduction

Recall that a regular semigroup S is Clifford if all idempotents of S are central. A semigroup S is called quasi-regular if for any a in S there exists a nature number m such that a^m is a regular element of S. A quasi-regular semigroup S is called a Clifford quasi-regular semigroup if S is a semilattice of quasi-groups and $Reg(S)$ is an ideal of S (see [4]). A semigroup S with zero is called nil if for any $a \in S$ there exists a nature number n such that $a^n = 0$. If S is a semigroup and K is an ideal of S such that the Rees quotient semigroup S/K is isomorphic to a semigroup Q, then we say that S is an ideal extension of K by Q. Furthermore, when Q is a nil semigroup, S is called a nil-extension of K by Q. It was shown in [4] that a nil-extension of a Clifford semigroup K by a nil semigroup Q is Clifford quasi-regular.

The class of Clifford semigroups play a fundamental role in the development of semigroup theory. It was proved by Clifford that a semigroup S is a Clifford semigroup if and only if S is a semilattice of groups; or if and only if S is a strong semilattice of groups. In 1994, Ren-Guo-Shum have already studied Clifford quasi-regular semigroup in [4]. Also, the congruence on completely quasi-regular semigroups has been described by Shum-Guo- Ren [1] by using admissible congruence pairs.

Here we shall study congruences on a Clifford quasi-regular semigroup S. It is proved that every congruence σ on such a semigroup S can be uniquely represented by a congruence pair (σ_Q, σ_K) on S, where σ_K is the restriction of σ on K, $\sigma_Q = (\sigma \vee \sigma_K)/\rho_K$ and ρ_K is the Rees congruence on S induced by an ideal K.

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Throughout this paper, S is a Clifford quasi-regular semigroup. We use ρ_K to denote the Rees congruence on S induced by an ideal K of S. Denote the set of all regular elements of S by $Reg(S)$ and the set of all idempotents of S by $E(S)$. H_a^* denote the $\mathcal{H}^*$-class of S containing a, where $\mathcal{H}^*$ be a generalized Green's relation (see [6]). Let $\mathcal{C}(S)$ be the set of all congruences on S and let $A\sigma$ be the set $\{a \in S \mid (\exists x \in A)(a, x) \in \sigma\}$ for any $\sigma \in \mathcal{C}(S)$. The σ-class containing the zero element 0 is denoted by 0σ. For terminologies and notations not mentioned in this paper, the reader is referred to [2].

§2. Preliminaries

We shall first state some basic properties concerning congruences on a Clifford quasi-regular semigroup S.

Proposition 2.1. For any $a \in S$ there exists a unique idempotent $e \in E(S)$ such that $a \in H_e^*$. Moreover, $a^n \in H_e$ if $n \geqslant r(a)$, where $r(a) = min\{n \mid a^n \in Reg(S)\}$.

Proof. The proof follows directly from the definition.

Remark. For the sake of convenience, we use a° to denote the unique idempotent of $\mathcal{H}^*$-class of S containing a. It can be easily verified that $Reg(S) = K$.

Proposition 2.2. Let σ be any congruence on S. Then $a \in K\sigma$ if and only if $(a, aa^\circ) \in \sigma$ and $(a, a^\circ a) \in \sigma$.

Proof. Suppose that $a \in K\sigma$. Then it is obvious that $(a, aa^\circ) \in \sigma$ and $(a, a^\circ a) \in \sigma$ if $a \in K$. On the other hand, if $a \notin K$, then we can find an element $x \in K$ such that $(a, x) \in \sigma$. Since S is a nil-extension of a Clifford semigroup K, there exists a positive integer $n \in N$ such that $a^n \in K$. This implies that $a^n\sigma = x^n\sigma$. However, since $a^n\sigma \cdot (a^n)^{-1}\sigma = a^\circ\sigma$, where $(a^n)^{-1}$ is the group inverse of a^n, we have $(a^n\sigma)^{-1} = (a^n)^{-1}\sigma$. Thus, $(a^n)^{-1}\sigma = (a^n\sigma)^{-1} = (x^n\sigma)^{-1} = (x^n)^{-1}\sigma$. This means that $(a^\circ, x^\circ) \in \sigma$ so that $(aa^\circ, xx^\circ) = (aa^\circ, x) \in \sigma$ and $(a^\circ a, x^\circ x) = (a^\circ a, x) \in \sigma$. Hence $(a, aa^\circ) \in \sigma$ and $(a, a^\circ a) \in \sigma$ by the transitivity of σ. Conversely, if $(a, aa^\circ) \in \sigma$ and $(a, a^\circ a) \in \sigma$, then by the above remark, we have $aa^\circ \in K$ and $a^\circ a \in K$. This implies that $a \in K\sigma$.

Proposition 2.3. If σ is a congruence on S, then $(a^\circ, b^\circ) \in \sigma$ for any $(a, b) \in \sigma$.

Proof. This proof is the same as the proof of Proposition 2.3 in [1].

Lemma 2.4[4]**.** The following statements on a semigroup S are equivalent:

(i) S is a Clifford quasi-regular semigroup;

(ii)S is quasi-regular, $E(S)$ is in the center of S and $Reg(S)$ is an ideal of S;

(iii)S is a nil-extension of a Clifford semigroup.

§3. Congruence pairs

Let S be a nil-extension of a Clifford semigroup K by a nil semigroup Q. In order to obtain a description of any congruence on S, we introduce the following definition.

Definition 3.1. Suppose that δ is a congruence on a semigroup Q and ω is a congruence on a semigroup K. Then a pair $(\delta, \omega) \in \mathcal{C}(Q) \times \mathcal{C}(K)$ is called a congruence pair on S if it satisfy the following conditions

(M1) If $(e,f)\in\omega$ for some idempotents $e,f\in E(S)$, then $(pe,pf)\in\omega$ for any $p\in Q$. Dually, $(ep,fp)\in\omega$ for any $p\in Q$;

(M2) If $(p,q)\in\delta\mid_{Q\setminus 0\delta}$, then $(pe,qe)\in\omega$ and $(ep,eq)\in\omega$ for any $e\in E(S)$;

(M3) If $(p,q)\in\delta\mid_{Q\setminus 0\delta}$, then $((pc)^\circ,(qc)^\circ)\in\omega$ and $((cp)^\circ,(cq)^\circ)\in\omega$ for any $c\in S$;

(M4) If $0\neq a\in 0\delta$ and $c\in S$, then

(i) $(aa^\circ c,ac(ac)^\circ)\in\omega$ and dually $(caa^\circ,ca(ca)^\circ)\in\omega$,

(ii) $(a^\circ ac,(ac)^\circ ac)\in\omega$ and dually $(ca^\circ a,(ca)^\circ ca)\in\omega$;

(M5) If $(pe,qf)\in\omega$ for some idempotents $e,f\in E(S)$ and any $p,q\in S$, then $(pe,fq)\in\omega$. Dually, if $(ep,fq)\in\omega$, then $(ep,qf)\in\omega$;

(M6) If $(pe,fq)\in\omega$ for some idempotents $e,f\in E(S)$ and any $p,q\in S$, then $(ep,qf)\in\omega$. Dually, if $(ep,qf)\in\omega$, then $(pe,fq)\in\omega$.

Now suppose that S is a nil-extension of a Clifford semigroup K by a nil semigroup Q and ρ_K is the Rees congruence on S induced by the ideal K. For any $\sigma\in\mathcal{C}(S)$, we define a mapping $\Gamma:\sigma\mapsto(\sigma_Q,\sigma_K)$ from $\mathcal{C}(S)$ to $\mathcal{C}(Q)\times\mathcal{C}(K)$, where $\sigma_K=\sigma\mid_K$, and $\sigma_Q=(\sigma\vee\rho_K)/\rho_K$. Thus, in view of the above definition, we have the following result.

Lemma 3.2. Let $\mathcal{C}(S)$ be the set of all congruences on S. Then $\sigma\subseteq\tau$ if and only if $\sigma_Q\subseteq\tau_Q$ and $\sigma_K\subseteq\tau_K$, for any $\sigma,\tau\in\mathcal{C}(S)$.

Proof. This proof is the same as the proof of Theorem 3.2 in [1].

Lemma 3.3. If $\sigma\in\mathcal{C}(S)$, then (σ_Q,σ_K) is a congruence pair on S.

Proof. Let $\sigma\in\mathcal{C}(S)$. Clearly, $\sigma_Q\in\mathcal{C}(Q)$ and $\sigma_K\in\mathcal{C}(K)$. To see that (σ_Q,σ_K) is a congruence pair on S, we only need to verify that (σ_Q,σ_K) satisfies the conditions (M1) to (M6) given in Definition 3.1.

(i) If $(e,f)\in\sigma_K$ for some $e,f\in E(S)$, then for any $p\in Q$, we have $(pe,pf)\in\sigma$. It is easy to see that $pe,pf\in K$. Thereby, we have $(pe,pf)\in\sigma_K$. Similarly, $(ep,fp)\in\sigma_K$. Thus the condition (M1) is satisfied.

(ii) By using the similar arguments as in (i), we can also see that (σ_Q,σ_K) satisfies condition (M2).

(iii) If $(p,q)\in\sigma_Q\mid_{Q\setminus 0\sigma_Q}$, then by the definition of σ_Q, we have $(p,q)\in\sigma$ and hence $(pc,qc)\in\sigma$ for any $c\in S$. By Proposition 2.3, It can be immediately seen that $((pc)^\circ,(qc)^\circ)\in\sigma$. Consequently $((pc)^\circ,(qc)^\circ)\in\sigma_K$. Dually, $((cp)^\circ,(cq)^\circ)\in\sigma_K$. This shows that (σ_Q,σ_K) satisfies the condition (M3).

(iv) Let $a\in 0\sigma_Q\setminus\{0\}$. Then by Proposition 2.2, $(a,aa^\circ)\in\sigma$ and so $(ac,aa^\circ c)\in\sigma$ for any $c\in S$, that is, $ac\in K\sigma$. In this case, $(ac,ac(ac)^\circ)\in\sigma$. It follows that $(aa^\circ c,ac(ac)^\circ)\in\sigma$ and consequently $(aa^\circ c,ac(ac)^\circ)\in\sigma_K$. A similar argument can show that $(caa^\circ,ca(ca)^\circ)\in\sigma_K$. Thus, (σ_Q,σ_K) satisfies the condition (M4)(i). On the other hand, since $a\in 0\sigma_Q\setminus\{0\}$, $(a,a^\circ a)\in\sigma$ by Proposition 2.2. Hence, for any $c\in S$ $(ac,a^\circ ac)\in\sigma$ giving $ac\in K\sigma$. In this case, $(ac,(ac)^\circ ac)\in\sigma$. We have that $(a^\circ ac,(ac)^\circ ac)\in\sigma$ and $(ac,(ac)^\circ ac)\in\sigma_K$. The dual $(ca^\circ a,(ca)^\circ ca)\in\omega$ can be similarly proved. Thus, (σ_Q,σ_K) satisfied the condition (M4)(ii);

(v) Let $(pe,qf)\in\omega$ for any $p,q\in S$ and $e,f\in E(S)$. Then $qf=fq$ since S is a nil-extension of a Clifford semigroup and by Proposition 2.4. This shows that $(pe,fq)\in\omega$. The dual part can be similarly proved. Hence, (σ_Q,σ_K) satisfies the condition (M5);

(vi) A similar arguments can show that (σ_Q,σ_K) satisfies the condition (M6).

Thus, by Definition 3.1, (σ_Q, σ_K) is indeed a congruence pair on S.

Lemma 3.4. Let $\mathcal{C}(S)$ be the set of all congruences on S. Then $\sigma \subseteq \tau$ if and only if $\sigma_Q \subseteq \tau_Q$ and $\sigma_K \subseteq \tau_K$, for any $\sigma, \tau \in \mathcal{C}(S)$.

(1°) $(a, b) \in \delta$ for any $a, b \in S \setminus R$;

(2°) $(aa^\circ, b^\circ b) \in \omega$ for any $a, b \in R$, where $R = K \bigcup \{0\delta \setminus \{0\}\}$. Then σ is a congruence on S such that $K\sigma = R$.

Proof. We first prove that the relation σ defined above is an equivalence on S. Let $a \in R$, then $(aa^\circ, aa^\circ) \in \omega$. Also by the given condition(M5)(i), we have $(aa^\circ, a^\circ a) \in \omega$. Obviously, the above relation σ is reflexive. To show that the relation σ is symmetric, let $a, b \in R$ such that $(aa^\circ, b^\circ b) \in \omega$. Then by (M6), $(a^\circ a, bb^\circ) \in \omega$. Also since $\omega \in \mathcal{C}(K)$, $(bb^\circ, a^\circ a) \in \omega$. It is easy to see that the above relation σ is symmetric. In order to show that the relation σ is transitive. Let $a, b \in R$ such that $(aa^\circ, b^\circ b) \in \omega$ and $(bb^\circ, c^\circ c) \in \omega$. Then $(b^\circ b, b^\circ b) \in \omega$ by $\omega \in \mathcal{C}(K)$. And since the given condition (M5)(i), we have $(b^\circ b, bb^\circ) \in \omega$. Thus $(aa^\circ, c^\circ c) \in \omega$. It implies that the above relation σ satisfies transitive. Hence, the relation σ is an equivalence. We now proceed to prove that σ is a congruence on S.

(i) Suppose that $a, b \in S \setminus R$ such that $(a, b) \in \sigma$ if and only if $(a, b) \in \delta$. Then we have either $ac \in S \setminus R$ or $ac \in R$ for any $c \in S$. If $ac \in S \setminus R$ then $bc \in S \setminus R$ since δ is a congruence on Q. This implies that $(ac, bc) \in \delta$ and so $(ac, bc) \in \sigma$. If $ac \in R$, then by the definition of δ, we know that $bc \in R$. If $c \in K$, then $ac, bc \in K$. But from $(a, b) \in \delta$, for any $a, b \in S \setminus R$ and by the condition (M2) of a congruence pair, we can easily observe that $(ac^\circ, bc^\circ) \in \omega$. As $\omega \in C(K)$, we immediately note that $(ac^\circ c, bc^\circ c) = (ac, bc) = (ac(ac)^\circ, (bc)^\circ bc) \in \omega$. This shows that $(ac, bc) \in \sigma$ whenever $c \in K$.

It remains to show that $(ac, bc) \in \sigma$ when $c \notin K$ and $(a, b) \in \delta$. In fact, if $c \in Q$, then $((ac)^\circ, (bc)^\circ) \in \omega$ by the given condition (M3). Thus, by the condition (M1), we have

$$(c(ac)^\circ, c(bc)^\circ) \in \omega. \tag{1}$$

Furthermore, we observe that, by Proposition 2.3, $((c(ac)^\circ)^\circ, (c(bc)^\circ)^\circ) \in \omega$. Thus, by the conditions (M1) and (M2), we have $(b(c(ac)^\circ)^\circ, b(c(bc)^\circ)^\circ) \in \omega$ and $(a(c(ac)^\circ)^\circ, b(c(ac)^\circ)^\circ) \in \omega$, whence $(a(c(ac)^\circ)^\circ, b(c(bc)^\circ)^\circ) \in \omega$. Together with the obtained property (3.1), we obtain that

$$(a(c(ac)^\circ)^\circ c(ac)^\circ, b(c(bc)^\circ)^\circ c(bc)^\circ) = (ac(ac)^\circ, bc(bc)^\circ) \in \omega.$$

By the given condition (M5)(i), we have $(ac(ac)^\circ, (bc)^\circ bc) \in \omega$. It is trivial to see that $(ac, bc) \in \omega$ by using condition (2°).

(ii) Suppose that $(aa^\circ, b^\circ b) \in \omega$ for $a, b \in R$. We now show that $(ac, bc) \in \sigma$ for any $c \in S$. In fact, by Proposition 2.3, $((aa^\circ)^\circ, (b^\circ b)^\circ) \in \omega$. Thus, for any $c \in S$, $((aa^\circ)^\circ c, (b^\circ b)^\circ c) \in \omega$ by the condition (M1) and hence $(aa^\circ c, b^\circ bc) \in \omega$ by assumption. If $a \in K$, then it is trivial to see that $aa^\circ c = ac$ and $(ac, ac(ac)^\circ) \in \omega$. Consequently, $(aa^\circ c, ac(ac)^\circ) \in \omega$. If $a \in 0\delta \setminus \{0\}$, then $(aa^\circ c, ac(ac)^\circ) \in \omega$ by the condition (M4)(i). Also if $b \in K$, then $b^\circ bc = bc$ and $(bc, (bc)^\circ bc) \in \omega$. Consequently, $(b^\circ bc, (bc)^\circ bc) \in \omega$. If $b \in 0\delta \setminus \{0\}$, then $(b^\circ bc, (bc)^\circ bc) \in \omega$ by the condition (M4)(ii). Thus, by the transitivity of congruence ω, we may deduce that $(ac(ac)^\circ, (bc)^\circ bc) \in \omega$. This shows that $(ac, bc) \in \omega$ since the condition (2°) is satisfied by the pair (ac, bc). Thus, σ is a left congruence on S.

Likewise, we can show that σ is also a right congruence on S and hence σ is a congruence on S. Furthermore, it is easy to see that $K\sigma = R$.

By using Lemma 3.4, the following theorem for congruence pairs on S is established.

Lemma 3.5. Let S be a nil-extension of a Clifford semigroup K by a nil semigroup Q and let (δ, ω) be a congruence pair on S. Then a congruence σ given in Lemma 3.4 is a unique congruence on S satisfying $\sigma_Q = \delta$ and $\sigma_K = \omega$.

Proof. We first show that $\sigma_Q = \delta$. To see that $\delta \subseteq \sigma_Q$, we let $a, b \in Q$ such that $(a, b) \in \delta$. Then, we have $a, b \in Q \setminus R$ or $a, b \in 0\delta \setminus \{0\}$ in Q. If $a, b \in Q \setminus R$, then $(a, b) \in \sigma_Q$ if and only if $(a, b) \in \delta$ by the definition of σ. On the other hand, if $a, b \in 0\delta \setminus \{0\}$, then by the definition of σ, we have $a, b \in K\sigma$ and so $a, b \in 0\sigma_Q$. This show that $\delta \subseteq \sigma_Q$ and so $\delta = \sigma_Q$ since $K\sigma = R$.

We still need to show that $\sigma_K = \omega$. For this purpose, we pick $a, b \in K$ such that $(a, b) \in \omega$. Then, it is trivial to see that $(aa^\circ, b^\circ b) \in \omega$. It follows from the definition of σ that $(a, b) \in \sigma_K$. Conversely, if $(a, b) \in \sigma_K$ for $a, b \in K$, then $(a, b) \in \sigma$. Thereby, $(a, b) = (aa^\circ, b^\circ b) \in \omega$ and hence $\sigma_K = \omega$. Finally, by using the facts given in theorem 3.2, we can observe that the congruence σ satisfying $\sigma_Q = \delta$ and $\sigma_K = \omega$ must be unique.

Summarizing the above results, we obtain the following theorem.

Theorem 3.6. S be a nil-extension of a Clifford semigroup K by a nil semigroup Q. Then a mapping $\Gamma : \sigma \longmapsto (\sigma_Q, \sigma_K)$ is an order-preserving bijection from the set of all congruences on S onto the set of all congruence pairs on S.

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Sequences of numbers in generalized arithmetic and geometric progressions

A.A.K. Majumdar

Beppu-shi 875-8577, Oita-ken, Japan
majumdar@apu.ac.jp aakmajumdar@gmail.com

Abstract In a recent paper, Zhang and Zhang [1] introduced the concept of sequences of numbers with alternate common differences. In this paper, we extend the idea to sequences in geometric progression. We also revisit some of the results of Zhang and Zhang to provide simpler and shorter forms and proofs in some cases.

Keywords Sequence of numbers with alternate common differences, periodic sequence with two common differences, sequence of numbers with alternate common ratio, periodic sequence with two common ratios, the general term and the sum of a sequence of numbers.

§1. Introduction

In a recent paper, Zhang and Zhang [1] introduced the concept of sequences of numbers in arithmetic progression with alternate common differences as well as the periodic sequence with two common differences. In this paper, we extend the concept to sequences of numbers in geometric progression with alternate common ratios and the periodic sequence with two common ratios. In doing so, we revisit the results of Zhang and Zhang [1], in Section 2 and Section 4 respectively. We give some of the results and their proofs in simpler forms. In Section 3, the sequence of numbers with two common ratios is treated. We derive the explicit forms of the general term, a_n, and the sum of first n terms S_n of the sequence. Section 5 deals with the periodic sequence of numbers in geometric progression with two common ratios.

§2. Sequence of numbers with alternate common differences

The sequence of numbers with alternate common differences, defined by Zhang and Zhang [1], is as follows.

Definition 2.1. A sequence of numbers a_n is called one with alternate common differences if the following two conditions are satisfied:

(1) For all $k \in \mathbf{N}$, $a_{2k} - a_{2k-1} = d_1$;

(2) For all $k \in \mathbf{N}$, $a_{2k+1} - a_{2k} = d_2$,

where d_1 and d_2 are two fixed numbers, called respectively the first common difference and the

second common difference of the sequence.

Lemma 2.1. Let $\{a_n\}$ be a sequence of numbers with alternate common differences d_1 and d_2. Then, for all $k \geqslant 1$,

$$a_{2k-1} = a_1 + (k-1)(d_1 + d_2), \tag{1}$$

$$a_{2k} = a_1 + kd_1 + (k-1)d_2. \tag{2}$$

Proof. From Definition 2.1, adding the two conditions therein,

$$a_{2i+1} - a_{2i-1} = d_1 + d_2$$

for all $i \geqslant 1$. Then,

$$\sum_{i=1}^{k-1} (a_{2i+1} - a_{2i-1}) = (k-1)(d_1 + d_2),$$

that is, $a_{2k-1} - a_1 = (k-1)(d_1 + d_2)$,

so that, $a_{2k-1} = a_1 + (k-1)(d_1 + d_2)$,

$$a_{2k} = a_{2k-1} + d_1 = a_1 + kd_1 + (k-1)d_2.$$

Corollary 2.1. Let $\{a_n\}$ be a sequence of numbers with alternate common differences d_1 and d_2. Then,

$$a_n = a_1 + \left[\frac{n}{2}\right] d_1 + \left[\frac{n-1}{2}\right] d_2,$$

where $[x]$ denotes the greatest integer less than or equal to x.

Proof. If n is odd, say, $n = 2k - 1$ for some integer $k \geq 1$, then,

$$\left[\frac{n-1}{2}\right] = k - 1 = \left[\frac{n}{2}\right]$$

and (1) can be expressed as

$$a_{2k-1} = a_n = a_1 + (k-1)d_1 + (k-1)d_2 = a_1 + \left[\frac{n}{2}\right] d_1 + \left[\frac{n-1}{2}\right] d_2.$$

And if n is even, say, $n = 2k$ for some integer $k \geqslant 1$, then

$$\left[\frac{n}{2}\right] = k, \quad \left[\frac{n-1}{2}\right] = k - 1,$$

so that (2) can be rewritten as

$$a_{2k} = a_n = a_1 + kd_1 + (k-1)d_2 = a_1 + \left[\frac{n}{2}\right] d_1 + \left[\frac{n-1}{2}\right] d_2.$$

Lemma 2.2. Let $\{a_n\}$ be a sequence of numbers with alternate common differences d_1 and d_2, and let $\{S_n\}$ be the sequence of $n-th$ partial sums. Then, for all $k \geqslant 1$,

$$S_{2k-1} = (2k-1)a_1 + (k-1)\{kd_1 + (k-1)d_2\}, \tag{3}$$

$$S_{2k} = 2ka_1 + k\{kd_1 + (k-1)d_2\}. \tag{4}$$

Proof. By definition,

$$S_{2k} = \sum_{i=1}^{2k} a_i = \sum_{i=1}^{k} (a_{2i-1} + a_{2i}).$$

Using Lemma 2.1, $S_{2k} = \sum_{i=1}^{k} \{2a_1 + (2i-1)d_1 + 2(i-1)d_2\} = 2ka_1 + k^2d_1 + k(k+1)d_2$, so that $S_{2k-1} = S_{2k} - a_{2k} = (2k-1)a_1 + k(k-1)d_1 + (k-1)^2d_2$.

Corollary 2.2. Let $\{a_n\}$ be a sequence of numbers with alternate common differences d_1 and d_2. Then,

$$S_n = na_1 + \left[\frac{n}{2}\right]\left(\left[\frac{n+1}{2}\right]d_1 + \left[\frac{n-1}{2}\right]\right).$$

Proof. If $n = 2k-1$ for some integer $k \geqslant 1$, then,

$$\left[\frac{n}{2}\right] = k-1 = \left[\frac{n-1}{2}\right], \quad \left[\frac{n+1}{2}\right] = k,$$

so that, from (3),

$$\begin{aligned} S_{2k-1} = S_n &= (2k-1)a_1 + (k-1)\{kd_1 + (k-1)d_2\} \\ &= na_1 + \left[\frac{n}{2}\right]\left(\left[\frac{n+1}{2}\right]d_1 + \left[\frac{n-1}{2}\right]d_2\right), \end{aligned}$$

and if $n = 2k$ for some integer $k \geqslant 1$, then,

$$\left[\frac{n}{2}\right] = k = \left[\frac{n+1}{2}\right], \quad \left[\frac{n-1}{2}\right] = k-1,$$

and from (4),

$$\begin{aligned} S_{2k} = S_n &= ka_1 + k\{kd_1 + (k-1)d_2\} \\ &= na_1 + \left[\frac{n}{2}\right]\left(\left[\frac{n+1}{2}\right]d_1 + \left[\frac{n-1}{2}\right]d_2\right). \end{aligned}$$

§3. Sequence of numbers with alternate common ratios

The sequence of numbers with alternate common ratios is defined as follows.

Definition 3.1. A sequence of numbers $\{a_n\}$ is called one with alternate common ratios if the following two conditions are satisfied :

(1) For all $k \in \mathbf{N}$, $\dfrac{a_{2k}}{a_{2k-1}} = r_1$,

(2) For all $k \in \mathbf{N}$, $\dfrac{a_{2k+1}}{a_{2k}} = r_2$,

where r_1 and r_2 are two fixed numbers, called respectively the first common ratio and the second common ratio of the sequence.

Lemma 3.1. Let $\{a_n\}$ be a sequence of numbers with alternate common ratios r_1 and r_2. Then, for all $k \geqslant 1$,

(1) $a_{2k-1} = a_1(r_1r_2)^{k-1}$;

(2) $a_{2k} = a_1 r_1^k r_2^{k-1}$.

Proof. From Definition 3.1,

$$\frac{a_{2k+1}}{a_{2k-1}} = \frac{a_{2k}}{a_{2k-1}} \cdot \frac{a_{2k+1}}{a_{2k}} = r_1r_2 \tag{5}$$

for all integer $k \geq 1$. Therefore,

$$\frac{a_{2k-1}}{a_{2k-3}} \cdot \frac{a_{2k-3}}{a_{2k-5}} \cdots \frac{5}{3} \cdot \frac{3}{1} = (r_1r_2) \cdot (r_1r_2) \cdots (r_1r_2) = (r_1r_2)^{k-1},$$

so that

$$a_{2k-1} = a_1(r_1r_2)^{k-1}.$$

Then,

$$a_{2k} = r_1 a_{2k-1} = a_1 r_1^k r_2^{k-1}.$$

Corollary 3.1. Let $\{a_n\}$ be a sequence of numbers with alternate common ratios r_1 and r_2. Then, both the subsequences $\{a_{2n-1}\}$ and $\{a_{2n}\}$ are sequences of numbers with common ratio r_1r_2.

Proof. From (5), we see that $\{a_{2n-1}\}$ is in geometric progression with common ratio r_1r_2. Again, since

$$\frac{a_{2k+2}}{a_{2k}} = \frac{a_{2k+2}}{a_{2k+1}} \cdot \frac{a_{2k+1}}{a_{2k}} = r_1r_2$$

for all integer $k \geq 1$, it follows that $\{a_{2n}\}$ is also in GP with common ratio r_1r_2.

Lemma 3.2. Let $\{a_n\}$ be a sequence of numbers with alternate common ratios r_1 *and* r_2, and let S_n be the sequence of n-th partial sums. Then, for all $n \geq 1$,

(1) $S_{2n-1} = \dfrac{a_1}{1 - r_1r_2}\left[1 - (r_1r_2)^n + r_1\{1 - (r_1r_2)^{n-1}\}\right]$;

(2) $S_{2n} = \dfrac{a_1}{1 - r_1r_2}\left[1 - (r_1r_2)^n\right](1 + r_1)$.

Proof. By definition,

$$S_{2n-1} = \sum_{i=1}^{2n-1} a_i = \sum_{i=1}^{n} a_{2i-1} + \sum_{i=1}^{n-1} a_{2i}.$$

Using Lemma 3.1,

$$\begin{aligned} S_{2n-1} &= \sum_{i=1}^{n} a_1 (r_1\, r_2)^{i-1} + \sum_{i=1}^{n-1} a_1 r_1^i\, r_2^{i-1} \\ &= a_1 \frac{1-(r_1\, r_2)^{n-1}}{1-r_1\, r_2} + a_1 r_1 \frac{1-(r_1\, r_2)^{n-1}}{1-r_1\, r_2}, \end{aligned}$$

which gives the desired expression for S_{2n-1} after some algebraic manipulation.
Then,

$$\begin{aligned} S_{2n} &= S_{2n-1} + a_{2n} \\ &= \frac{a_1}{1-r_1 r_2}\left[1-(r_1 r_2)^n + r_1\{1-(r_1 r_2)^{n-1}\}\right] + a_1 r_1^n r_2^{n-1} \\ &= \frac{a_1}{1-r_2 r_2}\left[1-(r_1 r_2)^n\right](1+r_1). \end{aligned}$$

Lemma 3.3. Let $\{a_n\}$ be a sequence of numbers with alternate common ratios r_1 and r_2 with $|r_1 r_2| < 1$. Then, the infinite series $\sum\limits_{n=1}^{\infty} a_n$ is convergent, and

$$\sum_{n=1}^{\infty} a_n = a_1 \frac{1+r_1}{1-r_1 r_2}.$$

Proof. From Lemma 3.2, both the sequences $\{S_{2n-1}\}_{n=1}^{\infty}$ and $\{S_{2n}\}_{n=1}^{\infty}$ are convergent with

$$\lim_{n\to\infty} S_{2n-1} = a_1 \frac{1+r_1}{1-r_1\, r_2} = \lim_{n\to\infty} S_{2\mathrm{n}}.$$

§4. Periodic sequence of numbers with two common differences

The periodic sequence of numbers with two common differences has been defined by Zhang and Zhang [1] as follows.

Definition 4.1. A sequence of numbers $\{a_n\}$ is called periodic with period p and two common differences if the following two conditions are satisfied :

(1) For all $k = 1, 2, 3, \cdots$,

$$a_{(k-1)p+1}, a_{(k-1)p+2}, \cdots, a_{kp}$$

is a finite arithmetic progression with common difference d_1;

(2) For all $k = 1, 2, 3, \cdots$,

$$a_{kp+1} = a_{kp} + d_3,$$

where $p > 1$ is a fixed integer, and d_1 and d_3 are two fixed numbers.

Zhang and Zhang [1] found the expressions for a_n and S_n for this sequence, which are rather complicated. We derive the expressions for a_n and S_n for this sequence under the assumption that

$$d_3 = d_1 + d_2. \tag{6}$$

These are given in Lemma 4.4 and Lemma 4.5 respectively. Note that the assumption (6) is not restrictive : Given the numbers d_1 and d_3, we can always find the number d_2 satisfying the condition (6).

Following Zhang and Zhang [1], we shall say that the terms $\{a_1, a_2, \ldots, a_p\}$ belong to the 1^{st} interval of length p, the terms $\{a_{p+1}, a_{p+2}, \ldots, a_{2p}\}$ belong to the 2^{nd} interval of length p, and so on, and in general, the terms $\{a_{(k-1)p+1}, a_{(k-1)p+2}, \ldots, a_{kp}\}$ belong to the $k-th$ interval of length p. Note that, in each interval, the terms are in arithmetic progression (AP) with common difference d_1. Thus, in particular, in the $k-th$ interval,

$$a_{kp} = a_{(k-1)p+1} + (p-1)d_1. \tag{7}$$

Lemma 4.1. Let $\{a_n\}$ be a p-periodic sequence of numbers with two common differences d_1 and d_2. Then, for all $k = 1, 2, 3, \ldots,$

$$a_{kp} = a_1 + (kp-1)d_1 + (k-1)d_2.$$

Proof. The proof is by induction on k . The result is clearly true for $k = 1$. So, we assume the validity of the result for some integer $k > 1$. Now,

$$a_{(k+1)p} = a_{kp+1} + (p-1)d_1.$$

But,

$$a_{kp+1} = a_{kp} + d_1 + d_2.$$

Therefore, using the induction hypothesis, we get

$$\begin{aligned} a_{(k+1)p} &= (a_{kp} + d_1 + d_2) + (p-1)d_1 \\ &= \{a + (kp-1)d_1 + (k-1)d_2\} + pd_1 + d_2 \\ &= a_1 + \{(k+1)p - 1\}d_1 + kd_2, \end{aligned}$$

which shows that the result is true for $k+1$ as well.

Lemma 4.2. Let $\{a_n\}$ be a p-periodic sequence of numbers with two common differences d_1 and d_2. Let

$$(k-1)p + 1 \leq \ell \leq kp$$

for some $k \in \{1, 2, 3, \ldots\}$. Then,

$$a_\ell = a_1 + (\ell - 1)d_1 + (k-1)d_2.$$

Proof. Since $(k-1)p + 1 \leq \ell \leq kp$, it follows that

$$a_\ell = a_{(k-1)p+1} + [\ell - (k-1)p - 1]\, d_1.$$

But, by Lemma 4.1,

$$\begin{aligned} a_{(k-1)p+1} &= a_{(k-1)p} + d_1 + d_2 \\ &= [a_1 + \{(k-1)p - 1\}d_1 + (k-2)d_2] + d_1 + d_2 \\ &= a_1 + (k-1)pd_1 + (k-1)d_2. \end{aligned}$$

Therefore, we finally get

$$\begin{aligned} a_\ell &= \{a_1 + (k-1)pd_1 + (k-1)d_2\} + \{\ell - (k-1)p - 1\}d_1 \\ &= a_1 + (\ell-1)d_1 + (k-1)d_2. \end{aligned}$$

Lemma 4.3. Let $(k-1)p + 1 \le \ell \le kp$ for some $k \in \{1, 2, 3, \cdots\}$ $(p \ge 2)$. Then,

$$k = \left[\frac{\ell - 1}{p}\right] + 1.$$

Proof. From the given inequality, we get

$$k - 1 \le \frac{\ell - 1}{p}, \quad k \ge \frac{\ell}{p}.$$

Since k, p and ℓ are all (positive) integers, it follows that

$$k - 1 = \left[\frac{\ell - 1}{p}\right].$$

Lemma 4.4. Let $\{a_n\}$ be a p-periodic sequence of numbers with two common differences d_1 and d_2 with $p \ge 2$. Then,

$$a_n = a_1 + (n-1)d_1 + \left[\frac{n-1}{p}\right] d_2.$$

Proof. follows immediately from Lemma 4.2 and Lemma 4.3.

Lemma 4.5. Let $\{a_n\}$ be a p-periodic sequence of numbers with two common differences d_1 and d_2 with $p \ge 2$, and let $\{S_n\}$ be the sequence of $n - th$ partial sums. Then,

$$S_n = na_1 + \frac{n(n-1)}{2}d_1 + \left[\frac{n-1}{p}\right]\left\{n - \frac{p}{2}\left(\left[\frac{n-1}{p}\right] + 1\right)\right\} d_2.$$

Proof. By definition,

$$\begin{aligned} S_n &= \sum_{i=1}^{n} a_i \\ &= \sum_{i=1}^{n} \left\{a_1 + (i-1)d_1 + \left[\frac{i-1}{p}\right] d_2\right\} \\ &= na_1 + \frac{n(n-1)}{2}d_1 + d_2 \sum_{i=1}^{n} \left[\frac{i-1}{p}\right]. \end{aligned}$$

Now, to calculate $\sum_{i=1}^{n} \left[\frac{i-1}{p}\right]$, let $(k-1)p + 1 \le n \le kp$ for some $k \in \{1, 2, 3, \cdots\}$. Then, the sum $\sum_{i=1}^{n} \left[\frac{i-1}{p}\right]$ can be written as

$$\sum_{i=1}^{n} \left[\frac{i-1}{p}\right] = \sum_{i=1}^{k-1} \sum_{j=(i-1)p+1}^{ip} \left[\frac{j-1}{p}\right] + \sum_{i=(k-1)p+1}^{\mathrm{n}} \left[\frac{i-1}{p}\right].$$

To find the above sums, note that

$$\left[\frac{j-1}{p}\right] = i-1$$

for all $(i-1)p+1 \leq j \leq ip$.
Therefore,

$$\begin{aligned}\sum_{i=1}^{n}\left[\frac{i-1}{p}\right] &= \sum_{i=1}^{k-1} p(i-1) + \{n-(k-1)p\}(k-1)\\ &= p\frac{(k-2)(k-1)}{2} + \{n-(k-1)p\}(k-1)\\ &= n(k-1) - \frac{p}{2}k(k-1)\\ &= (k-1)\left\{n - \frac{p}{2}k\right\}.\end{aligned}$$

Therefore,

$$S_n = na_1 + \frac{n(n-1)}{2}d_1 + (k-1)\left\{n-\frac{p}{2}k\right\}d_2.$$

We then get the desired result by virtue of Lemma 4.3.

If $p=1$, from Lemma 4.4 and Lemma 4.5, we see that the terms of sequence $\{a_n\}$ are in AP with common difference d_1+d_2. Also, recall that, if the terms of a sequence $\{b_n\}$are in AP with common difference d_1, then $b_n = d_1+(n-1)d_1$, and the sum of first n of its terms is simply $na_1 + \frac{n(n-1)}{2}d_1$. These expressions may be compared with those given in Lemma 4.4 and Lemma 4.5.

§5. Periodic sequence of numbers with two common ratios

The periodic sequence of numbers with two common ratios is defined as follows.

Definition 5.1. A sequence of numbers $\{a_n\}$ is called periodic with period p and two common ratios if the following two conditions are satisfied :

(1) For all $k = 1, 2, 3, \cdots$, the subsequence

$$\{a_{(k-1)p+1}, a_{(k-1)p+2}, \ldots, a_{kp}\}$$

is a finite geometric progression with common ratio r_1;

(2) For all $k = 1, 2, 3, \cdots$, $\frac{a_{kp+1}}{a_{kp}} = r_1 r_2$,

where $p > 1$ is a fixed integer, and r_1 and r_2 are two fixed numbers.

As in Section 4, the terms $\{a_1, a_2, \cdots, a_p\}$ belong to the 1^{st} interval of length p, the terms $\{a_{p+1}, a_{p+2}, \cdots, a_{2p}\}$ belong to the 2^{nd} interval of length p, and so on, and in general, the terms $\{a_{(k-1)p+1}, a_{(k-1)p+2}, \cdots, a_{kp}\}$ belong to the $k-th$ interval of length p. Note that, in each interval, the terms are in geometric progression (GP) with common ratio r_1. Thus, in particular, in the k -th interval,

$$a_{kp} = a_{(k-1)p+1} r_1^{p-1}. \tag{8}$$

Lemma 5.1. Let$\{a_n\}$ be a p-periodic sequence of numbers with two common ratios r_1 and r_2. Then, for all $k = 1, 2, 3, \cdots$,

(1) $a_{kp} = a_1 r_1^{kp-1} r_2^{k-1}$;
(2) $a_{kp+1} = a_1 r_1^{kp} r_2^{k}$.

Proof. To prove part (1), note that the result is true $k = 1$. To proceed by induction, we assume its validity for some integer $k > 1$. Now,

$$a_{(k+1)p} = a_{kp+1} r_1^{p-1}.$$

But,

$$a_{kp+1} = a_{kp} r_1 r_2.$$

Therefore, using the induction hypothesis, we get

$$\begin{aligned} a_{(k+1)p} &= (a_{kp} r_1 r_2) r_1^{p-1} \\ &= (a_1 r_1^{kp-1} r_2^{k-1})(r_1 r_2) r_1^{p-1}) \\ &= a_1 r_1^{(k+1)p-1} r_2^{k} \end{aligned}$$

showing the validity of the result for $k + 1$.
Now,

$$a_{kp+1} = a_{kp} r_1 r_2 = (a_1 r_1^{kp-1} r_2^{k-1})(r_1 r_2) = a_1 r_1^{kp} r_2^{k}.$$

This establishes part (2) of the lemma.

Lemma 5.2. Let $\{a_n\}$ be a p-periodic sequence of numbers with two common ratios r_1 and r_2. Let

$$(k-1)p + 1 \le n \le kp$$

for some $k \in \{1, 2, 3, \cdots\}$. Then,

$$a_n = a_1 r_1^{n-1} r_2^{k-1}.$$

Proof. Since $(k-1)p + 1 \le n \le kp$, it follow that

$$a_n = a_{(k-1)p+1} r_1^{n-(k-1)p-1}.$$

But, by part (2) of Lemma 5.1

$$a_{(k-1)p+1} = a_1 r_1^{(k-1)p} r_2^{k-1}.$$

Hence,

$$a_n = \left\{ a_1 r_1^{(k-1)p} r_2^{k-1} \right\} r_1^{n-(k-1)p-1} = a_1 r_1^{n-1} r_2^{k-1}.$$

Lemma 5.3. Let $\{a_n\}$ be a p-periodic sequence of numbers with two common ratios r_1 and r_2. Let $\{S_n\}$ be the sequence of $n - th$ partial sums. Let

$$(k-1)p + 1 \le n \le kp$$

for some $k \in \{1, 2, 3, \cdots\}$. Then,

$$S_n = \frac{a_1}{1 - r_1} \left(\frac{1 - r_1^p}{1 - r_1^p r_2} \left\{ 1 - (r_1^p r_2)^{k-1} \right\} + (r_1^p r_2)^{k-1} \{ 1 - r_1^{n-(k-1)p-1} \} \right).$$

Proof. By definition,

$$S_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{k-1} \sum_{j=(i-1)p+1}^{ip} a_j + \sum_{i=(k-1)p+1}^{n} a_i.$$

Now, note that

$$\sum_{j=(i-1)p+1}^{ip} a_j = a_{(i-1)p+1} \left(\frac{1-r_1^p}{1-r_1} \right),$$

$$\sum_{i=(k-1)p+1}^{n} a_i = a_{(k-1)p+1} \left(\frac{1-r_1^{n-(k-1)p-1}}{1-r_1} \right),$$

the expression for S_n takes the form

$$S_n = \frac{1-r_1^p}{1-r_1} \left(\sum_{i=1}^{k-1} a_{(i-1)p+1} \right) + a_{(k-1)p+1} \left(\frac{1-r_1^{n-(k-1)p-1}}{1-r_1} \right).$$

By part (2) of Lemma 5.1,

$$\sum_{i=1}^{k-1} a_{(i-1)p+1} = a_1 \sum_{i=1}^{k-1} (r_1^p r_2)^{i-1} = a_1 \left(\frac{1-(r_1^p r_2)^{k-1}}{1-r_1^p r_2} \right).$$

Therefore,

$$S_n = a_1 \left(\frac{1-r_1^p}{1-r_1} \right) \left(\frac{1-(r_1^p r_2)^{k-1}}{1-r_1^p r_2} \right) + a_1 (r_1^p r_2)^{k-1} \left(\frac{1-r_1^{n-(k-1)p-1}}{1-r_1} \right).$$

§6. Some remarks

From the proofs of Corollary 2.1 and Corollary 2.2, we have

$$\left[\frac{n-1}{2} \right] + \left[\frac{n}{2} \right] = n-1$$

for any integer $n \geq 1$,

$$\left[\frac{n}{2} \right] = \frac{n}{2} \Rightarrow \left[\frac{n-1}{2} \right] + \left[\frac{n+1}{2} \right] = n-1$$

for any integer $n \geq 1$,

$$\left[\frac{n}{2} \right] = \frac{n-1}{2} \Rightarrow [\frac{n-1}{2}] + \left[\frac{n+1}{2} \right] = n$$

for any integer $n \geq 1$.

Therefore, in the particular case when $d_1 = d_2 = d$, say, so that the sequence of numbers $\{a_n\}$ forms an AP with common difference d, the expressions of a_n and S_n, given respectively in Corollary 2.1 and Corollary 2.2, take the following forms :

$$a = a_1 + \left[\frac{n}{2} \right] d_1 + \left[\frac{n-1}{2} \right] d_2 = a_1 + (n-1)d,$$

$$S_n = na_1 + \left[\frac{n}{2}\right]\left(\left[\frac{n+1}{2}\right]d_1 + \left[\frac{n-1}{2}\right]d_2\right) = na_1 + \frac{n(n-1)}{2}d.$$

These are well-known results.

Again, in the particular case when $r_1 = r_2 = r$, say, the sequence of numbers $\{a_n\}$ in Section 3 forms a GP with common ratio r. In this case, the expressions for S_{2n-1} and S_{2n}, given in Lemma 3.2, can be shown to reduce to the well-known forms. This is done below :

$$\begin{aligned} S_{2n-1} &= \frac{a_1}{1-r_1r_2}\left[1-(r_1r_2)^n + r_1\left\{1-(r_1r_2)^{n-1}\right\}\right] \\ &= \frac{a_1}{1-r^2}\left[1-r^{2n}+r\left\{1-r^{2(n-1)}\right\}\right] = \frac{a_1}{1-r^2}(1+r)(1-r^{2n-1}) \\ &= \frac{a_1}{1-r}(1-r^{2n-1}), \end{aligned}$$

$$S_{2n} = \frac{a_1}{1-r_1r_2}\left[1-(r_1r_2)^n\right](1+r_1) = \frac{a_1}{1-r^2}(1-r^{2n})(1+r) = \frac{a_1}{1-r}(1-r^{2n}).$$

The p-periodic sequence of numbers with two common differences d_1 and d_2 has been treated in Section 4. Our approach is a little bit different from that of Zhang and Zhang [1]; however, to get the corresponding results given in Zhang and Zhang, one may put $d_2 = d_3 - d_1$ in Lemma 4.4 and Lemma 4.5. Clearly, in the particular case when $d_2 = 0$, the sequence an in Section 4 becomes one with common difference d_1.

Finally, we observe from Lemma 5.2 and Lemma 5.3 in Section 5 that, the terms of the sequence $\{a_n\}$ are in GP with common ratio r_1 in the particular case when $r_2 = 1$, and they are in GP with common ratio r_1r_2 in the particular case when $p = 1$. From Lemma 5.3, we see that the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent when $|r_1| < 1$ and $|r_2| < 1$, in such a case,

$$\sum_{n=1}^{\infty} a_n = \frac{a_1}{1-r_1}\left(\frac{1-r_1^p}{1-r_1^p\, r_2}\right).$$

It might be interesting to study the behavior of the infinite series $\sum_{n=1}^{\infty} a_n$, when $|r_1r_2| < 1$.

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Properties of bi γ -operations on topological spaces

B. Ahmad [†], T. Noiri [‡] and S. Hussain [#]

[†] Department of Mathematics, King Abdul Aziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia.
[‡] 2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumamoto-ken, 869-5142, Japan.
[#] Department of Mathematics, Islamia University, Bahawalpur, Pakistan.
E.mail: drbashir9@gmail.com
E.mail: t.noiri@nifty.com
E.mail: sabiriub@yahoo.com

Abstract In this paper, we continue studying the properties of (γ, γ')-operations on topological spaces initiated by J. Umehara, H. Maki and T. Noiri [12] and further investigated by S. Hussain and B. Ahmad [5].

Keywords γ-closed (open), γ-closure, γ-regular (open), $(\gamma, \gamma^{'})$-interior, (γ, γ')-exterior, $(\gamma, \gamma^{'})$ -boundary, $(\gamma, \gamma^{'})$-closure, $(\gamma, \gamma^{'})$-open (closed), $(\gamma, \gamma^{'})$-nbd, $(\gamma, \gamma^{'})$-derived set, $(\gamma, \gamma^{'})$-dense and $(\gamma, \gamma^{'})$-dense in itself.

§1. Introduction

In 1979, S. Kasahara [7] introduced the concept of α-closed graphs of a function. D. S. Jankovic [6] defined α-closed sets and studied functions with α-closed graphs. H. Ogata [9] introduced the notions of γ-$T_i, i = 0, 1/2, 1, 2$; (γ, β)- homeomorphism and studied some topological properties. B. Ahmad and F. U. Rehman ([1] , [11]) defined and investigated several properties of γ-interior, γ-exterior, γ-closure and γ-boundary points in topological spaces. They also discussed their properties in product spaces and studied the characterizations of (γ, β)- continuous mappings initiated by H. Ogata [10]. In 2003, 2005 and 2006, B.Ahmad and S.Hussain [2-4] continued studying the properties of γ-operations on topological spaces introduced by S. Kasahara [7]. They also defined and discussed several properties of γ-nbd, γ-nbd base at x, γ-closed nbd, γ-limit point, γ-isolated point, γ-convergent point and γ^*-regular space. They further defined γ- normal spaces, γ_0-compact [4] and established many interesting properties. In 1992, J. Umeraha, H. Maki and T. Noiri [12]; and in 1994, J. Umehara [13] defined and discussed the properties of $(\gamma, \gamma^{'})$-open sets, $(\gamma, \gamma^{'})$- closure, $(\gamma, \gamma^{'})$-generalized closed sets in a space X. In 2006, S. Hussain and B. Ahmad [5] continued to discuss the properties of $(\gamma, \gamma^{'})$-open sets, $(\gamma, \gamma^{'})$-closure, $(\gamma, \gamma^{'})$-generalized closed sets in a space X defined in [12].

In this paper, we continue studying properties of $(\gamma, \gamma^{'})$- operations on topological spaces. Hereafter we shall write spaces in place of topological spaces.

§2. Preliminaries

In order to make this paper self-contained, we give the preliminaries used in the sequel.

Definition 2.1[10]. Let (X, τ) be a space. An operation $\gamma : \tau \to P(X)$ is a function from τ to the power set of X such that $V \subseteq V^{\gamma}$, for each $V \in \tau$, where V^{γ} denotes the value of γ at V. The operations defined by $\gamma(G) = G$, $\gamma(G) = cl(G)$ and $\gamma(G) = int(cl(G))$ are examples of operation γ.

Definition 2.2[10]. Let $A \subseteq X$. A point $a \in A$ is said to be a γ-interior point of A, if there exists an open nbd N of a such that $N^{\gamma} \subseteq A$. We denote the set of all such points by int_{γ}(A).Thus
$int_{\gamma}(A) = \{x \in A : x \in N \in \tau \ \ and \ \ N^{\gamma} \subseteq A\} \subseteq A$.

Note that A is γ-open [10] iff $A = int_{\gamma}(A)$. A set A is called γ- closed [10] iff $X - A$ is γ-open.

Definition 2.3[10]. An operation γ on τ is said to be regular, if for any open nbds U, V of $x \in X$, there exists an open nbd W of x such that $U^{\gamma} \cap V^{\gamma} \supseteq W^{\gamma}$.

Definition 2.4[10]. An operation γ on τ is said to be open, if for any open nbd U of each $x \in X$, there exists an γ-open set B such that $x \in B$ and $U^{\gamma} \supseteq B$.

Definition 2.5[5]. Let A be a subset of X. A point $x \in A$ is said to be $(\gamma, \gamma^{'})$-interior point of A iff there exist open nbds U and V of x such that $U^{\gamma} \cup V^{\gamma^{'}} \subseteq A$.
We define the set of all such points as $(\gamma, \gamma^{'})$- interior of A and is denoted as by $int_{(\gamma,\gamma')}(A)$.
Thus $int_{(\gamma,\gamma')}(A) = \{x \in A : \ x \in U, \ x \in V, U, V \in \tau$ and $U^{\gamma} \cup V^{\gamma^{'}} \subseteq A\} \subseteq A$.
If $A \subseteq B$, then $int_{(\gamma,\gamma')}(A) \subseteq int_{(\gamma,\gamma')}(B)$.

Definition 2.6[12]. A subset A of (X, τ) is said to be a $(\gamma, \gamma^{'})$-open set iff $int_{(\gamma,\gamma')}(A) = A$.

Note that the class of all $(\gamma, \gamma^{'})$-open sets of (X, τ) is denoted by $\tau_{(\gamma,\gamma')}$.

§3. Properties of bi γ-operations

Definition 3.1. Let X be a space and $x \in X$. Then a $(\gamma, \gamma^{'})$-nbd of x is a set U which contains a $(\gamma, \gamma^{'})$-open set V containing x. Evidently, a set U is a $(\gamma, \gamma^{'})$-nbd of x if $x \in int_{(\gamma,\gamma')}(U)$. The class of all $(\gamma, \gamma^{'})$-nbds of x is called the $(\gamma, \gamma^{'})$-nbd system at x and is denoted by U_x.

Theorem 3.1. The $(\gamma, \gamma^{'})$-nbd system U_x at x in a space X has the following properties:
(1) If $U \in U_x$, then $x \in U$.
(2) If $U, V \in U_x$, then $U \cap V \in U_x$, where γ and $\gamma^{'}$ are regular operations.
(3) If $U \in U_x$, then there is a $V \in U_x$ such that $U \in U_y$, then each $y \in V$.
(4) If $U \in U_x$ and $U \subseteq V$, then $V \in U_x$
(5)(a) If $U \subseteq X$ is $(\gamma, \gamma^{'})$-open, then U contains a $(\gamma, \gamma^{'})$-nbd of each of its points.
(b)If U contains a $(\gamma, \gamma^{'})$-nbd of each of its points, then U is $(\gamma, \gamma^{'})$-open, provided γ and $\gamma^{'}$ are regular.

Proof. (1) is obvious.
(2) If $U, V \in U_x$, then $x \in int_{(\gamma,\gamma')}(U)$ and $x \in int_{(\gamma,\gamma')}(V)$ imply $x \in int_{(\gamma,\gamma')}(U) \cap int_{(\gamma,\gamma')}(V)$. Since γ and $\gamma^{'}$ are regular operations, therefore $x \in int_{(\gamma,\gamma')}(U \cap V)$. This implies $U \cap V \in U_x$.

(3) Let $U \in U_x$. Take $V = int_{(\gamma,\gamma')}(U)$. Then each $y \in V$ gives $y \in int_{(\gamma,\gamma')}(U)$ and therefore $U \in U_y$.
(4) If $U \in U_x$, then $x \in int_{(\gamma,\gamma')}(U)$. If $U \subseteq V$, then $int_{(\gamma,\gamma')}(U) \subseteq int_{(\gamma,\gamma')}(V)$ gives $x \in int_{(\gamma,\gamma')}(V)$. Hence $V \in U_x$.
(5) (a) If U is (γ,γ')-open, then $U = int_{(\gamma,\gamma')}(U)$ and therefore U is a (γ,γ')-nbd of each of its points.
(b) If $x \in U$ has a (γ,γ')-nbd $V_x \subseteq U$, then $U = int_{(\gamma,\gamma')}(\bigcup_{x\in U} V_x)$ is union of (γ,γ')-open sets and therefore is (γ,γ')-open. This completes the proof.
Combining 5(a) and 5(b) of the Theorem 3.1, we have

Proposition 3.1. A set $U \subseteq X$ is (γ,γ')-open iff U contains a (γ,γ')-nbd of each of its points, where γ and γ' are regular operations.

Definition 3.2. A (γ,γ')-nbd base at x in a space X is a subcollection β_x taken from (γ,γ')-nbd system U_x having the property that $U \in U_x$ contains some
$V \in \beta_x$. That is U_x must be determined by β_x as

$$U_x = \{U \subseteq X / V \subseteq U \quad for \quad some \quad V \in \beta_x\}.$$

Then each $U \in \beta_x$ is called a basic (γ,γ')-nbd of x.

Example 3.1. In any space X, the (γ,γ')-open nbds of x form a (γ,γ')-nbd base at x, since for any (γ,γ')-nbd U of x $int_{(\gamma,\gamma')}(U)$ is also a (γ,γ')-nbd of x.

Example 3.2. If X is the discrete space, then each $x \in X$ has a (γ,γ')-nbd base consisting of a single set, namely $\{x\}$, that is, $U_x = \{\{x\}\}$.

Theorem 3.2. Let X be a space and for each $x \in X$, let β_x be a (γ,γ')-nbd base at x. Then
(1) If $V \in \beta_x$, then $x \in V$.
(2) If $V_1, V_2 \in \beta_x$, then there is some $V_3 \in \beta_x$ such that $V_3 \subseteq V_1 \cap V_2$, where γ and γ' are regular operations.
(3) If $V \in \beta_x$, then there is some $V_3 \in \beta_x$ such that $y \in V$, then there is some $W \in \beta_x$ with $W \subseteq V$.
(4) (a) If $U \subseteq X$ is (γ,γ')-open, then U contains a basic (γ,γ')-nbd of each of its points.
(b) If U contains a basic (γ,γ')-nbd of each of its points, then U is (γ,γ')-open provided γ and γ' are regular operations.

Proof. Follows from Theorem 3.1.

Definition 3.3[12]. A point $x \in X$ is called a (γ,γ')-closure point of $A \subseteq X$, if $(U^{\gamma} \cup V^{\gamma'}) \cap A \neq \emptyset$, for any open nbds U and V of x. The set of all (γ,γ')-closure points is called the (γ,γ')-closure of A and is denoted by $cl_{(\gamma,\gamma')}(A)$.
Note that $cl_{(\gamma,\gamma')}(A)$ is contained in every (γ,γ')-closed superset of A. Clearly, a subset A of X is called (γ,γ')-closed, if $cl_{(\gamma,\gamma')}(A) \subseteq A$.

Theorem 3.3. Let X be a space. Suppose a (γ,γ')-nbd base has been fixed at each $x \in X$. Then
(1) $V \subseteq X$ is (γ,γ')-closed iff each point $x \notin V$ has a basic (γ,γ')-nbd disjoint from V.
(2) $cl_{(\gamma,\gamma')}(E) = \{x \in X$: each basic (γ,γ')-nbd of x meets $E\}$.

(3) $int_{(\gamma,\gamma')}(E) = \{x \in X$: each basic (γ,γ')-nbd of x contained in $E\}$.
(4) $bd_{(\gamma,\gamma')}(E) = \{x \in X$: each basic (γ,γ')-nbd of x meets both E and $X - E\}$.

Proof. (1) Follows from Theorem 3.1 (5).

(2) We know that $cl_{(\gamma,\gamma')}(E) = \cap\{K \subseteq X : K \ \ is(\gamma,\gamma')$-closed and $E \subseteq K\}$ Suppose some (γ,γ')-nbd U of x does not meet E, then $x \in int_{(\gamma,\gamma')}(U)$ and $E \subseteq X - int_{(\gamma,\gamma')}(U)$. Since $X - int_{(\gamma,\gamma')}(U)$ is (γ,γ')-closed, therefore $cl_{(\gamma,\gamma')}(E) \subseteq X - int_{(\gamma,\gamma')}(U)$[5]. Hence $x \notin cl_{(\gamma,\gamma')}(E)$.

Conversely, if $x \notin cl_{(\gamma,\gamma')}(E)$, then X-$cl_{(\gamma,\gamma')}(E)$ is a (γ,γ')-open set containing x and hence containing a basic (γ,γ')-nbd of x, which does not meet E. This completes the proof.

(3) This follows from (2) by an application of De Morgan's Laws on $int_{(\gamma,\gamma')}(E) = X$ -$cl_{(\gamma,\gamma')}(E)$[5].

(4) Follows from Theorem 3.1 (5) and the definition of $bd_{(\gamma,\gamma')}(E)$. This completes the proof.

Definition 3.4. A space X is said to be a (γ,γ')-T_1 space, if for any two distinct points x, y of X there exist open sets U and V containing x and y respectively such that $y \notin U^\gamma$ and $x \notin V^{\gamma'}$.

Theorem 3.4. A space X is a (γ,γ')-T_1 space iff each singleton in X is (γ,γ')-closed.

Proof. Let X be a (γ,γ')-T_1 space. We show that each singleton $\{x\}$ is (γ,γ')-closed. For this, we show that its complement $X - \{x\}$ is (γ,γ')-open. Let $y \in X - \{x\}$, $y \neq x$, then there exist sets U and V containing x and y respectively such that $y \notin U^\gamma$ and $x \notin V^{\gamma'}$. So $x \in U^\gamma, y \notin U^\gamma$ and $y \in V^\gamma, x \notin V^{\gamma'}$. This implies that $y \in U^\gamma \cup V^{\gamma'} \subseteq X - \{x\}$. So, $X - \{x\}$ is (γ,γ')-open. Hence $\{x\}$ is (γ,γ')-closed.

Conversely, suppose that $\{x\}$ is (γ,γ')-closed. We show that X is a (γ,γ')-T_1 space. Let $x, y \in X, x \neq y$. Then $X - \{x\}, X - \{y\}$ are (γ,γ')-open in X. Thus $x \in X - \{y\}$ and $y \in X - \{x\}$ such that $x \notin (X - \{x\})^\gamma$ and $y \notin (X - \{y\})^\gamma$. This completes the proof

Theorem 3.5. A space X is (γ,γ')-T_1 iff every finite subset of X is (γ,γ')-closed, where $\gamma$and γ' are regular operations.

Proof. Suppose that X is a (γ,γ')-T_1 space. Then by above Theorem 3.4, one point subsets of X are (γ,γ')-closed. Since γ and γ' are regular operations, so every finite subset of X, being a union of a finite number of (γ,γ')-closed sets, is (γ,γ')-closed.

Conversely, suppose that every finite subset of X is (γ,γ')-closed. Then every one point subset of X is (γ,γ')-closed. Hence X is a (γ,γ')-T_1 space. This completes the proof.

Theorem 3.6. In a (γ,γ')-T_1 space, each subset A of X is the intersection of its (γ,γ')-open supersets, where γ and γ' are regular operations.

Proof. Since in a (γ,γ')-T_1 space, each finite subset A of X is (γ,γ')-closed, where γ and γ' are regular operations. Since each singleton of X is (γ,γ')-closed, therefore each $y \in X - A, X - \{y\}$ is (γ,γ')-open and $A \subseteq X - \{y\}$. So that $X - \{y\}$ is a (γ,γ')-open superset of A. Since for each $x \in A, x \in X - \{y\}, A = \bigcap_{x \in A} (X - \{y\})$. This completes the proof.

Definition 3.5. Let X be a space and $x \in X$. Then x is called a (γ,γ')-limit point of A, if $(U^\gamma \cup V^{\gamma'}) \cap (A - \{x\}) \neq \emptyset$, where U, V are open sets in X.

Definition 3.6[4]. An operation $\gamma : \tau \to P(X)$ is said to be strictly regular, if for any open nbds U, V of $x \in X$, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma = W^\gamma$.

Theorem 3.7. Let X be a (γ,γ')-T_1 space and $A \subseteq X$. If X is a (γ,γ')-limit point of A, then every (γ,γ')-open set containing x contains infinite number of distinct points of A, where γ and γ' are strictly regular operations.

Proof. Let U, W be (γ,γ')-open sets and $x \in U$ and $x \in W$. Suppose U as well as W contains only a finite number of distinct points of A. Then
$U \cap A = \{x_1, x_2, \cdots x_n\} = B_1$,
$W \cap A = \{y_1, y_2, \cdots y_n\} = B_2$.
Clearly B_1 is (γ,γ')-closed implies $(B_1)'$ is (γ,γ')-open ,Where $(B_1)'$ denotes the complement of B_1 and hence open . Similarly $U \cap \beta_1^{\gamma}$ is (γ,γ')-open and hence open. But $U \cap (B_1)'$ contains x and

$$\begin{aligned}(U \cap (B_1)')^{\gamma} \cap A &= (U^{\gamma} \cap ((B_1)')^{\gamma} \cap A \\ &= (U^{\gamma} \cap A) \cap (((B_1)')^{\gamma} \cap A) \subseteq (U \cap A) \cap ((B_1)' \cap A) \\ &= B_1 \cap ((B_1)' \cap A) \\ &= (B_1 \cap (B_1)') \cap A = \emptyset \cap A = \emptyset.\end{aligned}$$

Also B_2 is (γ,γ')-closed implies $(B_2)'$ is (γ,γ')-open and hence open. So $W \cap (B_2)'$ contains x. Similarly $(W \cap (B_2)')^{\gamma'} \cap A = \emptyset$.

Thus $((U \cap (B_1)')^{\gamma} \cap A) \cup ((W \cap (B_2)')^{\gamma'} \cap A) = \emptyset$. This implies that

$$((U \cap (B_1)')^{\gamma} \cup ((W \cap (B_2)')^{\gamma'}) \cap A = \emptyset$$

This shows that x is not a (γ,γ')-limit point of A, a contradiction. Thus U as well as W contains infinite number of distinct points of A. This completes the proof.

Corollary 3.1. In a (γ,γ')-T_1 space no finite subset has a (γ,γ')-limit point, where γ and γ' are strictly regular operations.

Definition 3.7. The set of a (γ,γ')-limit point of A, denoted $A^d_{(\gamma,\gamma')}$ is called (γ,γ')-derived set. It is easily seen that if $A \subseteq B$ then

$A^d_{(\gamma,\gamma')} \subseteq B^d_{(\gamma,\gamma')}$. $\qquad (*)$

Theorem 3.8. In a (γ,γ')-T_1 space the (γ,γ')-derived set has the following properties:

(1) $cl_{(\gamma,\gamma')}(A) = A \cup A^d_{(\gamma,\gamma')}$.

(2)$(A \cup B)^d_{(\gamma,\gamma')} = A^d_{(\gamma,\gamma')} \cup B^d_{(\gamma,\gamma')}$. In general

(3) $\bigcup_i (A_i)^d_{(\gamma,\gamma')} = (\bigcup_i A_i)^d_{(\gamma,\gamma')}$.

(4)$(A^d_{(\gamma,\gamma')})^d_{(\gamma,\gamma')} \subseteq A^d_{(\gamma,\gamma')}$.

(5) $cl_{(\gamma,\gamma')}(A^d_{(\gamma,\gamma')}) = A^d_{(\gamma,\gamma')}$.

Proof. (1) Let $x \in cl_{(\gamma,\gamma')}(A)$. Then $x \in C$, for every (γ,γ')-closed superset C of A. Now

(i) If $x \in A$, then $x \in A^d_{(\gamma,\gamma')}$.

(ii) If $x \notin A$, then we prove that $x \in A \cup A^d_{(\gamma,\gamma')}$. To prove (ii), suppose U and W are open sets containing x. Then $(U^{\gamma} \cup W^{\gamma'}) \cap A = \emptyset$, for otherwise $A \subseteq X - (U^{\gamma} \cap W^{\gamma'}) = C$, where C is a (γ,γ')-closed superset of A not containing x. This contradicts the fact that x belongs to every (γ,γ')-closed superset C of A. Therefore $x \in A^d_{(\gamma,\gamma')}$ gives $x \in A \cup A^d_{(\gamma,\gamma')}$.

Conversely, suppose that $x \in A \cup A^d_{(\gamma,\gamma')}$, we show that $x \in cl_{(\gamma,\gamma')}(A)$. If $x \in A$, then $x \in cl_{(\gamma,\gamma')}(A)$. If $x \in A^d_{(\gamma,\gamma')}$, then we show that x is in every (γ,γ')-closed superset of A. We suppose otherwise that there are (γ,γ')-closed supersets C_1 and C_2 of A not containing x. Then $x \in X - C_1 = U^\gamma$ and $x \in X - C_2 = W^{\gamma'}$ which are (γ,γ')-open and $U^\gamma \cap A = \emptyset$, $W^{\gamma'} \cap A = \emptyset$. So $(U^\gamma \cap A) \cup (W^{\gamma'} \cap A) = \emptyset \Longrightarrow (U^\gamma \cup W^{\gamma'}) \cap A = \emptyset$. This implies that $x \in A^d_{(\gamma,\gamma')}$. This contradiction proves that $x \in cl_{(\gamma,\gamma')}(A)$. Consequently $cl_{(\gamma,\gamma')}(A) = A \cup A^d_{(\gamma,\gamma')}$. This proves (1).

(2) $(A \cup B)^d_{(\gamma,\gamma')} \subseteq A^d_{(\gamma,\gamma')} \cup B^d_{(\gamma,\gamma')}$. Let $x \in (A \cup B)^d_{(\gamma,\gamma')}$. Then $x \in cl_{(\gamma,\gamma')}(A \cup B) - \{x\}$ or $x \in cl_{(\gamma,\gamma')}(A - \{x\}) \cup cl_{(\gamma,\gamma')}(B - \{x\})$, implies $x \in cl_{(\gamma,\gamma')}(A - \{x\})$ or $x \in cl_{(\gamma,\gamma')}(B - \{x\})$. This gives $x \in A^d_{(\gamma,\gamma')}$ or $x \in B^d_{(\gamma,\gamma')}$. Therefore $x \in A^d_{(\gamma,\gamma')} \cup B^d_{(\gamma,\gamma')}$. This proves $(A \cup B)^d_{(\gamma,\gamma')} \subseteq A^d_{(\gamma,\gamma')} \cup B^d_{(\gamma,\gamma')}$. The converse follows directly from (*).

(3) The proof is immediate from (*).

(4) Suppose that $x \notin A^d_{(\gamma,\gamma')}$. Then $x \notin cl_{(\gamma,\gamma')}(A - \{x\})$. This implies that there are open sets U and V such that $x \in U$ and $x \in V$ and $(U^\gamma \cup V^{\gamma'}) \cap (A - \{x\}) = \emptyset$. We prove that $x \notin (A^d_{(\gamma,\gamma')})^d_{(\gamma,\gamma')}$. Suppose on the contrary that $x \in (A^d_{(\gamma,\gamma')})^d_{(\gamma,\gamma')}$. Then $x \in cl_{(\gamma,\gamma')}(A^d_{(\gamma,\gamma')} - \{x\})$. Since $x \in U$ and $x \in V$, so $(U^\gamma \cup V^{\gamma'}) \cap (A^d_{(\gamma,\gamma')} - \{x\}) \neq \emptyset$. Therefore there is a $q \neq x$ such that $q \in (U^\gamma \cup V^{\gamma'}) \cap (A^d_{(\gamma,\gamma')})$. It follows that $q \in ((U^\gamma \cup V^{\gamma'}) - \{x\}) \cap (A^d_{(\gamma,\gamma')} - \{x\})$. Hence $((U^\gamma \cup V^{\gamma'}) - \{x\}) \cap (A^d_{(\gamma,\gamma')} - \{x\}) \neq \emptyset$, a contradiction to the fact that $(U^\gamma \cup V^{\gamma'}) \cap (A - \{x\}) = \emptyset$. This implies that $x \notin (A^d_{(\gamma,\gamma')})^d_{(\gamma,\gamma')}$ and so $(A^d_{(\gamma,\gamma')})^d_{(\gamma,\gamma')} \subseteq A^d_{(\gamma,\gamma')}$.

(5) This is a consequence of (1), (2) and (4). This completes the proof.

Definition 3.8. Let A be a subset of a space X and $A^d_{(\gamma,\gamma')}$ the set of all (γ,γ')-limit points of A . If $A \subseteq A^d_{(\gamma,\gamma')}$, then A is called (γ,γ')-dense in itself.

In a (γ,γ')-T_1 space, set (γ,γ')-dense in itself has the following properties:

Theorem 3.9. (1) If $\{A_i\}$ is an arbitrary family of subsets of a (γ,γ')-T_1 space X (γ,γ')-dense in themselves, then $\bigcup_i A_i \subseteq (\bigcup_i A_i)^d_{(\gamma,\gamma')}$, that is, $\bigcup_i A_i$ is (γ,γ')-dense in itself.

(2) (i) For any subset A of X,
$(cl_{(\gamma,\gamma')}(A))^d_{(\gamma,\gamma')} = (A \cup A^d_{(\gamma,\gamma')})^d_{(\gamma,\gamma')} = A^d_{(\gamma,\gamma')} \cup (A^d_{(\gamma,\gamma')})^d_{(\gamma,\gamma')} \subseteq A^d_{(\gamma,\gamma')} \cup A^d_{(\gamma,\gamma')} = A^d_{(\gamma,\gamma')}$
or $(cl_{(\gamma,\gamma')}(A))^d_{(\gamma,\gamma')} \subseteq A^d_{(\gamma,\gamma')} \subseteq A \cup A^d_{(\gamma,\gamma')} = cl_{(\gamma,\gamma')}(A)$

(ii) For any subset A of X, if $A \subseteq (A)^d_{(\gamma,\gamma')}$, then $cl_{(\gamma,\gamma')}(A) \subseteq (cl_{(\gamma,\gamma')}(A))^d_{(\gamma,\gamma')}$.

(3) If A is (γ,γ')-closed and (γ,γ')-dense in itself and B does not contain a subset P which is (γ,γ')-dense in itself, then each (γ,γ')-T_1 space X is the union of sets A and B.

Proof. (1) Suppose $\{A_i\}$ is a family of subsets of a (γ,γ')-T_1 space X such that each A_i is (γ,γ')-dense in itself. Then $A_i \subseteq (A_i)^d_{(\gamma,\gamma')}$. By Theorem 3.8 (3),
$\bigcup_i A_i \subseteq \bigcup_i (A_i)^d_{(\gamma,\gamma')} = (\bigcup_i A_i)^d_{(\gamma,\gamma')}$. This proves (1).

(2) (i) By Theorem 3.8 (1, 2 and 4), we infer that
$(cl_{(\gamma,\gamma')}(A))^d_{(\gamma,\gamma')} = (A \cup A^d_{(\gamma,\gamma')})^d_{(\gamma,\gamma')} \subseteq A^d_{(\gamma,\gamma')} \cup (A^d_{(\gamma,\gamma')})^d_{(\gamma,\gamma')} \subseteq A^d_{(\gamma,\gamma')} \cup A^d_{(\gamma,\gamma')} = A^d_{(\gamma,\gamma')}$
or $(cl_{(\gamma,\gamma')}(A))^d_{(\gamma,\gamma')} \subseteq A^d_{(\gamma,\gamma')} \subseteq A \cup A^d_{(\gamma,\gamma')} \subseteq cl_{(\gamma,\gamma')}(A)$
or $(cl_{(\gamma,\gamma')}(A))^d_{(\gamma,\gamma')} \subseteq cl_{(\gamma,\gamma')}(A)$.

(ii) Since $A \subseteq cl_{(\gamma,\gamma')}(A)$, therefore $A^d_{(\gamma,\gamma')} \subseteq (cl_{(\gamma,\gamma')}(A))^d_{(\gamma,\gamma')}$. Now

$cl_{(\gamma,\gamma')}(A) = A \cup A^d_{(\gamma,\gamma')} = A^d_{(\gamma,\gamma')} \subseteq (cl_{(\gamma,\gamma')}(A))^d_{(\gamma,\gamma')}$ or $cl_{(\gamma,\gamma')}(A) \subseteq (cl_{(\gamma,\gamma')}(A))^d_{(\gamma,\gamma')}$. This proves (2).

(3) Let $A = \bigcup_i A_i$, for each A_i (γ,γ')-dense in itself. Then by Theorem 3.9 (1), A is (γ,γ')-dense in itself. By Theorem 3.9 (2) (ii), $cl_{(\gamma,\gamma')}(A)$ is (γ,γ')-dense in themselves and hence $cl_{(\gamma,\gamma')}(A) \subseteq A$. That is, A is (γ,γ')-closed. Clearly the set $X - A = B$, being disjoint from A, does not contain nonempty sets which are (γ,γ')-dense in themselves. This completes the proof.

Combining 2(i) and 2 (ii) in the above Theorem, we have:

Proposition 3.2. If $A \subseteq (A)^d_{(\gamma,\gamma')}$, then $cl_{(\gamma,\gamma')}(A) = (cl_{(\gamma,\gamma')}(A))^d_{(\gamma,\gamma')}$.

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Some notes on the paper "The mean value of a new arithmetical function"

Jin Zhang [†‡] and Pei Zhang [†]

[†]Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China
[‡] Department of Mathematics, Xi'an Normal School, Xi'an, Shaanxi, P.R.China

Abstract In reference [2], we used the elementary method to study the mean value properties of a new arithmetical function, and obtained two mean value formulae for it, but there exist some errors in that paper. The main purpose of this paper is to correct the errors in reference [2], and give two correct conclusions.

Keywords Smarandache multiplicative function, mean value, asymptotic formula.

§1. Introduction

For any positive integer n, we call an arithmetical function $f(n)$ as the Smarandache multiplicative function if for any positive integers m and n with $(m,\ n)=1$, we have $f(mn)=\max\{f(m),\ f(n)\}$. For example, the Smarandache function $S(n)$ and the Smarandache LCM function $SL(n)$ both are Smarandache multiplicative functions. In reference [2], we defined a new Smarandache multiplicative function $f(n)$ as follows: $f(1)=1$; If $n>1$, then $f(n)=\max\limits_{1\leq i\leq k}\{\dfrac{1}{\alpha_i+1}\}$, where $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the factorization of n into prime powers. Then we studied the mean value properties of $f(n)$, and proved two asymptotic formulae:

$$\sum_{n\leq x}f(n)=\frac{1}{2}\cdot x\cdot\ln\ln x+\lambda\cdot x+O\left(\frac{x}{\ln x}\right),\tag{1}$$

where λ is a computable constant.

$$\sum_{n\leq x}\left(f(n)-\frac{1}{2}\right)^2=\frac{1}{36}\cdot\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\cdot\sqrt{x}\cdot\ln\ln x+d\cdot\sqrt{x}+O\left(x^{\frac{1}{3}}\right),\tag{2}$$

where $\zeta(s)$ is the Riemann zeta-function, and d is a computable constant.

But now, we found that the methods and results in reference [2] are wrong, so the formulae (1) and (2) are not correct. In this paper, we shall improve the errors in reference [2], and obtain two correct conclusions. That is, we shall prove the following:

Theorem 1. For any real number $x>1$, we have the asymptotic formula

$$\sum_{n\leq x}f(n)=\frac{1}{2}\cdot x+O\left(x^{\frac{1}{2}}\right).$$

Theorem 2. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}\left(f(n)-\frac{1}{2}\right)^2) = \frac{1}{36}\cdot\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\cdot\sqrt{x}+O\left(x^{\frac{1}{3}}\right),$$

where $\zeta(n)$ is the Riemann zeta-function.

§2. Proof of the theorems

In this section, we shall using the elementary and the analytic methods to prove our Theorems. First we give following two simple Lemmas:

Lemma 1. Let A denotes the set of all square-full numbers. Then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{\substack{n\leq x\\ n\in A}} 1 = \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\cdot x^{\frac{1}{2}}+\frac{\zeta\left(\frac{2}{3}\right)}{\zeta(2)}\cdot x^{\frac{1}{3}}+O\left(x^{\frac{1}{6}}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Lemma 2. Let B denotes the set of all cubic-full numbers. Then for any real number $x > 1$, we have

$$\sum_{\substack{n\leq x\\ n\in B}} 1 = N\cdot x^{\frac{1}{3}}+O\left(x^{\frac{1}{4}}\right),$$

where N is a computable constant.

Proof. The proof of these two Lemmas can be found in reference [3].

Now we use these two simple Lemmas to complete the proof of our Theorems. In fact, for any positive integer $n > 1$, from the definition of $f(n)$ we have

$$\sum_{n\leq x} f(n) = f(1)+\sum_{\substack{n\leq x\\ n\in A}} f(n)+\sum_{\substack{n\leq x\\ n\in B}} f(n), \tag{3}$$

where A denotes the set of all square-full numbers. That is, $n > 1$, and for any prime p, if $p \mid n$, then $p^2 \mid n$. B denotes the set of all positive integers $n > 1$ with $n \notin A$. Note that $f(n) \ll 1$, from the definition of A and Lemma 1 we have

$$\sum_{\substack{n\leq x\\ n\in A}} f(n) = O\left(x^{\frac{1}{2}}\right). \tag{4}$$

$$\begin{aligned}\sum_{\substack{n\leq x\\ n\in B}} f(n) &= \sum_{\substack{n\leq x\\ n\in B}}\frac{1}{2} = \sum_{n\leq x}\frac{1}{2}-\sum_{\substack{n\leq x\\ n\in A}}\frac{1}{2}\\ &= \frac{1}{2}\cdot x+O\left(x^{\frac{1}{2}}\right).\end{aligned} \tag{5}$$

Now combining (3), (4) and (5) we may immediately get

$$\begin{aligned}\sum_{n\leq x} f(n) &= 1+\sum_{\substack{n\leq x\\ n\in A}} f(n)+\sum_{\substack{n\leq x\\ n\in B}} f(n)\\ &= \frac{1}{2}\cdot x+O\left(x^{\frac{1}{2}}\right).\end{aligned}$$

This proves Theorem 1.

Now we prove Theorem 2. From the definition of $f(n)$ and the properties of square-full numbers we have

$$\begin{aligned}\sum_{n\leq x}\left(f(n)-\frac{1}{2}\right)^2 &= \frac{1}{4}+\sum_{\substack{n\leq x\\ n\in A}}\left(f(n)-\frac{1}{2}\right)^2+\sum_{\substack{n\leq x\\ n\notin A}}\left(f(n)-\frac{1}{2}\right)^2\\ &= \frac{1}{4}+\sum_{\substack{n\leq x\\ n\in A}}\left(f(n)-\frac{1}{2}\right)^2. \qquad (6)\end{aligned}$$

where A also denotes the set of all square-full numbers. Let C denotes the set of all cubic-full numbers. Then from the properties of square-full numbers, Lemma 1 and Lemma 2 we have

$$\begin{aligned}\sum_{\substack{n\leq x\\ n\in A}}\left(f(n)-\frac{1}{2}\right)^2 &= \sum_{\substack{n\leq x\\ n\in A,\ f(n)=\frac{1}{3}}}\left(\frac{1}{3}-\frac{1}{2}\right)^2+\sum_{\substack{n\leq x\\ n\in C}}\left(f(n)-\frac{1}{2}\right)^2\\ &= \sum_{\substack{n\leq x\\ n\in A}}\left(\frac{1}{3}-\frac{1}{2}\right)^2-\sum_{\substack{n\leq x\\ n\in C}}\left(\frac{1}{3}-\frac{1}{2}\right)^2+O\left(\sum_{\substack{n\leq x\\ n\in C}}1\right)\\ &= \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\cdot x^{\frac{1}{2}}+O\left(x^{\frac{1}{3}}\right). \qquad (7)\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function.

Now combining (6) and (7) we have the asymptotic formula

$$\sum_{n\leq x}\left(f(n)-\frac{1}{2}\right)^2=\frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\cdot x^{\frac{1}{2}}+O\left(x^{\frac{1}{3}}\right).$$

This completes the proof of Theorem 2.

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On the Smarandache multiplicative sequence

Ling Li† ‡

† Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China

‡ Department of Basic Courses, Shaanxi Polytechnic Institute, Xianyang, Shaanxi, P.R.China

Abstract In this paper, we use the elementary method to study the properties of the Smarandache multiplicative sequence, and proved that some infinite series involving the Smarandache multiplicative sequence are convergent.

Keywords Smarandache multiplicative sequence, infinite series, convergent properties.

§1. Introduction and result

For any positive integer $m \geq 2$, let $1 < d_1 < d_2 < \cdots < d_m$ are m positive integers, then we define the Smarandache multiplicative sequence A_m as: If $d_1,\ d_2,\ \cdots,\ d_m$ are the first m terms of the sequence A_m, then $d_k > d_{k-1}$, for $k \geq m+1$, is the smallest number equal to $d_1^{\alpha_1} \cdot d_2^{\alpha_2} \cdots d_m^{\alpha_m}$, where $\alpha_i \geq 1$ for all $i = 1,\ 2,\ \cdots,\ m$. For example, the Smarandache multiplicative sequence A_2 (generated by digits 2, 3) is:

$$2,\ 3,\ 6,\ 12,\ 18,\ 24,\ 36,\ 48,\ 54,\ 72,\ 96,\ 108,\ 144,\ 162,\ 192,\ 216,\ \cdots\cdots .$$

The Smarandache multiplicative sequence A_3 (generated by digits 2, 3, 7) is:

$$2,\ 3,\ 7,\ 42,\ 84,\ 126,\ 168,\ 252,\ 294,\ 336,\ 378,\ 504,\ 588,\ 672,\ \cdots\cdots .$$

The Smarandache multiplicative sequence A_4 (generated by digits 2, 3, 5, 7) is:

$$2,\ 3,\ 5,\ 7,\ 210,\ 420,\ 630,\ 840,\ 1050,\ 1260,\ 1470,\ 1680,\ 1890,\ 2100,\ \cdots\cdots .$$

In the book "Sequences of Numbers Involved Unsolved Problems", Professor F.Smarandache introduced many sequences, functions and unsolved problems, one of them is the Smarandache multiplicative sequence, and he also asked us to study the properties of this sequence. About this problem, it seems that none had studied it yet, at least we have not seen any related papers before. The problem is interesting, because there are close relationship between the Smarandache multiplicative sequence and the geometric series. In this paper, we shall use the elementary method to study the convergent properties of some infinite series involving the Smarandache multiplicative sequence, and get some interesting results. For convenience, we use the symbol $a_m(n)$ denotes the n-th term of the Smarandache multiplicative sequence A_m. The main purpose of this paper is to study the convergent properties of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{a_m^s(n)}, \tag{1}$$

and prove the following:

Theorem. For any positive integer $m \geq 2$, let $1 < d_1 < d_2 < \cdots < d_m$ are m positive integers, and A_m denotes the Smarandache multiplicative sequence generated by $d_1,\ d_2,\ \cdots,\ d_m$. Then for any real number $s \leq 0$, the infinite series (1) is divergent; For any real number $s > 0$, the series (1) is convergent, and

$$\sum_{n=1}^{\infty} \frac{1}{a_m^s(n)} = \prod_{i=1}^{m} \frac{1}{d_i^s - 1} + \sum_{i=1}^{m} \frac{1}{d_i^s}.$$

From our Theorem we may immediately deduce the following two corollaries:

Corollary 1. Let A_2 be the Smarandache multiplicative sequence generated by 2 and 3, then we have the identity

$$\sum_{n=1}^{\infty} \frac{1}{a_2(n)} = \frac{4}{3}.$$

Corollary 2. Let A_3 be the Smarandache multiplicative sequence generated by 3, 4 and 5, then we have the identity

$$\sum_{n=1}^{\infty} \frac{1}{a_3(n)} = \frac{13}{20}.$$

Similarly, we can also introduce another sequence called the Smarandache additive sequence as follows: Let $1 \leq d_1 < d_2 < \cdots < d_m$ are m positive integers, then we define the Smarandache additive sequence D_m as: If $d_1,\ d_2,\ \cdots,\ d_m$ are the first m terms of the sequence D_m, then $d_k > d_{k-1}$, for $k \geq m+1$, is the smallest number equal to $\alpha_1 \cdot d_1 + \alpha_2 \cdot d_2 + \cdots\cdots + \alpha_m \cdot d_m$, where $\alpha_i \geq 1$ for all $i = 1,\ 2,\ \cdots,\ m$. It is clear that this sequence has the close relationship with the coefficients of the power series ($x^{d_1} + x^{d_2} + \cdots + x^{d_m} < 1$)

$$\sum_{n=1}^{\infty} \left(x^{d_1} + x^{d_2} + \cdots + x^{d_m}\right)^n = \frac{x^{d_1} + x^{d_2} + \cdots + x^{d_m}}{1 - x^{d_1} - x^{d_2} - \cdots - x^{d_m}}.$$

For example, the Smarandache additive sequence D_2 (generated by digits 3, 5) is:

3, 5, 8, 11, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, $\cdots\cdots$.

It is an interesting problem to study the properties of the Smarandache additive sequence.

§2. Proof of the theorem

In this section, we shall prove our Theorem directly. First note that for any positive integer $k > m$, we have

$$a_m(k) = d_1^{\alpha_1} \cdot d_2^{\alpha_2} \cdots d_m^{\alpha_m},$$

where $\alpha_i \geq 1$, $i = 1,\ 2,\ \cdots,\ m$. So for any real number $s > 1$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{a_m^s(n)} &= \sum_{i=1}^{m} \frac{1}{a_m^s(i)} + \sum_{\alpha_1=1}^{\infty} \sum_{\alpha_2=1}^{\infty} \cdots \sum_{\alpha_m=1}^{\infty} \frac{1}{(d_1^{\alpha_1} \cdot d_2^{\alpha_2} \cdots d_m^{\alpha_m})^s} \\ &= \sum_{i=1}^{m} \frac{1}{a_m^s(i)} + \left(\sum_{\alpha_1=1}^{\infty} \frac{1}{d_1^{\alpha_1 s}}\right) \cdot \left(\sum_{\alpha_2=1}^{\infty} \frac{1}{d_2^{\alpha_2 s}}\right) \cdots \left(\sum_{\alpha_m=1}^{\infty} \frac{1}{d_m^{\alpha_m s}}\right). \end{aligned} \tag{2}$$

It is clear that for any real number $s \leq 0$, the series $\sum_{\alpha_i=1}^{\infty} \frac{1}{d_i^{\alpha_i s}}$ is divergent, and for any real number $s > 0$, the series $\sum_{\alpha_i=1}^{\infty} \frac{1}{d_i^{\alpha_i s}}$ is convergent, and more

$$\sum_{\alpha_i=1}^{\infty} \frac{1}{d_i^{\alpha_2 s}} = \frac{1}{d_i^s - 1}.$$

So from (2) we know that the series $\sum_{n=1}^{\infty} \frac{1}{a_m^s(n)}$ is also convergent, and

$$\sum_{n=1}^{\infty} \frac{1}{a_m^s(n)} = \prod_{i=1}^{m} \frac{1}{d_i^s - 1} + \sum_{i=1}^{m} \frac{1}{d_i^s}.$$

This completes the proof of our Theorem.

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